

# Toric Calabi-Yau Fourfolds

## Duality Between N=1 Theories and Divisors that Contribute to the Superpotential<sup>\*</sup>

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### ABSTRACT

We study issues related to  $F$ -theory on Calabi-Yau fourfolds and its duality to heterotic theory for Calabi-Yau threefolds. We discuss principally fourfolds that are described by reflexive polyhedra and show how to read off some of the data for the heterotic theory from the polyhedron. We give a procedure for constructing examples with given gauge groups and describe some of these examples in detail. Interesting features arise when the local pieces are fitted into a global manifold. An important issue is how to compute the superpotential explicitly. Witten has shown that the condition for a divisor to contribute to the superpotential is that it have arithmetic genus 1. Divisors associated with the short roots of non-simply laced gauge groups do not always satisfy this condition while the divisors associated to all other roots do. For such a ‘dissident’ divisor we distinguish cases for which  $\chi(\mathcal{O}_D) > 1$  corresponding to an  $X$  that is not general in moduli (in the toric case this corresponds to the existence of non-toric parameters). In these cases the ‘dissident’ divisor  $D$  does not remain an effective divisor for general complex structure. If however  $\chi(\mathcal{O}_D) \leq 0$ , then the divisor is general in moduli and there is a genuine instability.

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# 1. Introduction

This paper is devoted to a number of issues pertaining to the compactification of F-theory on Calabi–Yau fourfolds. For fourfolds,  $X$ , of particular structure there are believed to be interesting dualities [1]

$$F[X] = \text{IIB}[B^X] = \text{Het}[Z^X, V^X] , \quad (1.1)$$

that relate F-theory on  $X$  to IIB string theory compactified on a (non Calabi–Yau) threefold  $B^X$  and also to a heterotic compactification on a Calabi–Yau threefold  $Z^X$  with vector bundle  $V^X$ . The relation between heterotic compactification on threefolds and Calabi–Yau fourfolds that these dualities entail is particularly interesting since it offers the hope of insight into the important but so far poorly understood  $(0, 2)$  compactifications of the heterotic string. Relation (1.1) suggests that heterotic theories on Calabi–Yau threefolds are, in some sense, classified by Calabi–Yau fourfolds. We will here concern ourselves largely with Calabi–Yau fourfolds that are themselves described by reflexive polyhedra. Thus a class of heterotic theories on Calabi–Yau threefolds are described by reflexive five dimensional polyhedra. The question that we seek to answer is to what extent we can read off the data for the heterotic theory from this polyhedron.

A basic question is how to read off the the threefold of the heterotic theory from the data. This is accomplished by observing that the dual polyhedron for the heterotic threefold is obtained via a certain projection, which we explain, of the dual polyhedron of the fourfold. This toric description is dual to the picture of the heterotic threefold in terms of a maximal degeneration of the fourfold. We will argue that, at least for the models we consider here, the relation between the fourfold and the heterotic threefold involves mirror symmetry in an interesting way:

$$\begin{aligned} F[X] &= \text{Het}[Z, V^X] \quad , \quad \tilde{X} = (\tilde{Z}, \mathbb{P}_1) \\ F[\tilde{X}] &= \text{Het}[\tilde{Y}, V^{\tilde{X}}] \quad , \quad X = (Y, \mathbb{P}_1) \end{aligned}$$

where tildes are used to denote mirror manifolds and  $X = (Y, \mathbb{P}_1)$ , for example, denotes that  $X$  is a fibration by a (Calabi–Yau) threefold  $Y$  fibered over a  $\mathbb{P}_1$ .

Another important question is how to compute explicitly the  $F$ -theory superpotential on a given fourfold and try to understand this as a superpotential for the corresponding heterotic theory. When we do this several problems arise. At this point we are unable to give a complete answer; we are presenting here various examples and related observations.

An immediate question that arises on computing a superpotential from the fourfold  $X$  is that the superpotential is a cubic function of the Kähler parameters of  $X$ , while the heterotic superpotential is a function of the volumes of the curves, and thus linear. The comparison makes sense (so far) only for smooth divisors  $D$  contributing to worldsheet instantons; curiously this question has not arisen in examples that have been previously considered owing to the fact that in these examples the cubic expression in the Kähler parameters is in fact trilinear. We discuss these previous calculations in Section 7 (see also Section 5.2). It seems that the resolution is to regard the volume of the elliptic fibre as an infinitesimal and in this limit the volumes of the divisors are indeed trilinear.

Other questions arise when we compare the  $F$ -theory superpotential corresponding to gauge groups, with a  $d = 3$  dimensional Yang-Mills theory, following [2,3]. Their work deals with divisors corresponding to gauge groups [4], arising as resolution of the Weierstrass model of the elliptic fourfold: each (irreducible) divisor mapping via the elliptic fibration to the same surface can be identified with a node in the extended Dynkin diagram of the group. A key point in their computation is that all (or none) of the irreducible divisors of a Dynkin diagram contribute to the superpotential. In Section 4 we consider explicit examples where “mixed configurations” occur, that is when some divisors contribute to the superpotential, and some do not (following the criterion of [5]). This happens when the gauge group is not simply laced ( $SO(\text{odd})$ ,  $Sp(n)$ ,  $G_2$ ,  $F_4$ ). A non simply laced group arises via the action of monodromy on a group that is simply laced. In such a case the divisor (two divisors for the case of  $F_4$ ) that arise through the identification of divisors of the simply laced group may have  $\chi(\mathcal{O}_D) \neq 1$ , violating the condition to contribute to the superpotential. We shall refer to such divisors as being dissident. In Section 6 we show that there are many such examples. If  $\chi(\mathcal{O}_D) > 1$ ,  $X$  turns out not to be general in moduli (in the toric case this correspond to the existence of non-toric parameters) and that the ‘dissident’ divisor  $D$  will not remain an effective divisor for general values of the complex moduli space. If  $\chi(\mathcal{O}_D) \leq 0$ , then the divisor will be general in moduli and we cannot reproduce Vafa’s computations. On the other hand we show that in the toric case  $h^{2,1}(X) > 0$ : this always leads to an interesting structure on the heterotic dual (see [6,7]).

Many of our observations will be recognisable to those who have developed a local description of the duality in terms of branes wrapping the singular fibres of the fourfold seen as an elliptic fibration over the threefold base  $B^X$ . On the other hand our observation is that interesting features arise precisely as a result of trying to fit the local pieces together into a global manifold. In particular there is a tendency for the gauge group of the effective theory to ‘grow’ since maintaining the fibration structure of the fourfold  $X$ , when we put together the local singularities, requires further resolution of these singularities.

The layout of this paper is the following: in §2 we gather together some expressions that compute the Hodge numbers of a Calabi–Yau fourfold  $X$ , corresponding to a reflexive polyhedron  $\nabla$ , as well as the arithmetic genus of its divisors in terms of the combinatoric properties of  $\nabla$ . We find also interesting expressions, whose significance we do not properly understand, that relate the arithmetic genera of the divisors of a manifold to the arithmetic genera of the divisors of the mirror. In §3 we introduce our basic model fourfold  $X$  and describe its structure. This model is perhaps as simple as one can have without taking a model that is completely trivial. The gauge group that we have is  $SU(2) \times G_2$ . In §4 we explain the structure of this model. We set out originally to construct a model with group  $SU(2) \times SU(3)$  however maintaining the fibration structure of our particular choice  $X$  required extending the  $SU(3)$  to  $G_2$  in a way that we explain in detail. We also show how to extend the gauge group by taking the degenerate fibers to have a more complicated structure. The new element here is the process by which we show how to divide the fans in such a way as to maintain the fibration. In §5 we analyze the Yukawa coupling and the structure of the Mori cone; for the Mori Cone we implement a procedure advocated in [8]. It is clear however that our implementation of this procedure leads to a cone that is too large. This is similar to a recent result in [9]. In §7 we discuss the computation of the superpotential and compare this calculation to that for previous examples.

## 2. Toric Preliminaries

We draw together here some essential results that we will need. Calabi–Yau fourfolds,  $X$ , for which the holonomy is  $SU(4)$  rather than a subgroup have a Hodge diamond whose top half is of the form

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & 0 & & 0 & \\
 & & 0 & & h^{11} & & 0 \\
 & 0 & & h^{21} & & h^{21} & 0 \\
 1 & & h^{31} & & h^{22} & & h^{31} & 1 .
 \end{array}$$

There is also a linear relation between the Hodge numbers

$$h^{22} = 2(22 + 2h^{11} + 2h^{31} - h^{21}) .$$

Thus there are three independent Hodge numbers and the Euler number is given in terms of the Hodge numbers by

$$\chi_E(X) = 6(8 + h^{11} + h^{31} - h^{21}) . \quad (2.1)$$

The three independent Hodge numbers  $h^{31}$ ,  $h^{21}$  and  $h^{11}$  may be determined directly from the Newton polyhedron,  $\Delta$ , and its dual,  $\nabla$ , (our convention is that the fan of  $X$  is that fan over the faces of  $\nabla$ ) via the expressions [10-12]

$$\begin{aligned}
 h^{31} &= \text{pts}(\Delta) - \sum_{\text{codim } \theta=1} \text{int}(\theta) + \sum_{\text{codim } \theta=2} \text{int}(\theta) \text{int}(\tilde{\theta}) - 6 \\
 h^{11} &= \text{pts}(\nabla) - \sum_{\text{codim } \tilde{\theta}=1} \text{int}(\tilde{\theta}) + \sum_{\text{codim } \tilde{\theta}=2} \text{int}(\tilde{\theta}) \text{int}(\theta) - 6 \\
 h^{21} &= \sum_{\text{codim } \theta=3} \text{int}(\tilde{\theta}) \text{int}(\theta)
 \end{aligned} \quad (2.2)$$

In these expressions  $\text{pts}(\Delta)$  denotes the number of lattice points in  $\Delta$ , while  $\theta$  runs over the faces of  $\Delta$ ,  $\text{int}(\theta)$  denotes the number of lattice points strictly interior to  $\theta$  and  $\tilde{\theta}$  denotes the face of  $\nabla$  dual to  $\theta$ . It is interesting to note that Batyrev[13] gives also an expression for the “physicist’s Euler number” in terms of the *volumes* of the faces of the polyhedra

$$\chi'_E = \sum_{\text{codim } \theta=2} \text{vol}(\theta) \text{vol}(\tilde{\theta}) - 2 \sum_{\text{codim } \theta=3} \text{vol}(\theta) \text{vol}(\tilde{\theta}) + \sum_{\text{codim } \theta=4} \text{vol}(\theta) \text{vol}(\tilde{\theta}) \quad (2.3)$$

in which the volumes are normalized such that the volume of a fundamental lattice simplex is unity (rather than  $1/d!$  in  $d$  dimensions). If the fourfold is nonsingular then these two expressions for the Euler number yield the same result. There are cases however of fourfolds  $X_{mnp}$  defined by (4.1) for which  $\chi'_E \neq \chi_E$  indicating that these fourfolds are singular (for all values of their parameters).

Each point  $q \in \nabla$  that is not interior to a facet (codimension one face) corresponds to a divisor  $D_q$  of  $X$ . Such a point  $q$  defines a face  $\tilde{\theta}_q$  of  $\nabla$ , the unique face to which it is interior<sup>1</sup>, and in virtue of duality also a face  $\theta_q$  of  $\Delta$ . Klemm *et al.* [12] establish an elegant expression for the arithmetic genus,  $\chi_q$ , of  $D_q$ :

$$\chi_q = \chi(\mathcal{O}(D_q)) = 1 - (-1)^{\dim(\tilde{\theta}_q)} \text{int}(\theta_q) . \quad (2.4)$$

An observation, made already in [12], is that manifolds containing divisors of arithmetic genus one are abundant. These exist whenever  $\nabla$  has faces that (a) have interior points and that (b) are dual to faces of  $\Delta$  that have no interior points. Klemm *et al.* study an interesting class of manifolds that we will term the KLRY spaces, and whose construction we review in Sect. 4. All these manifold and all their mirrors have such faces<sup>2</sup>.

Witten [5] shows that  $\chi(\mathcal{O}(D_q)) = 1$  is a necessary condition to contribute to the superpotential, while

$$h^0(\mathcal{O}(D_q)) = 1 , \quad h^i(\mathcal{O}(D_q)) = 0 , \quad i > 0$$

is a sufficient condition.

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<sup>1</sup> A point  $q \in \nabla$  that is not the interior point is contained in some face of  $\nabla$ . We ask whether  $q$  is interior to this face or if it lies in the boundary. If it lies in the boundary then it lies in a face of lower dimension and we ask again if it lies in the interior to this face or in the boundary. Proceeding in this way we either arrive at the unique face to which  $q$  is interior or we find that  $q$  is a vertex. It is therefore convenient to adopt the convention that the vertices are interior to themselves.

<sup>2</sup> The number of these divisors is by (2.4) always finite. The infinite number found in [14] is associated with the fact the manifold under consideration is not in the class that we consider here, since it cannot be realised as a hypersurface in a toric variety given by a single equation, and hence by a single polyhedron. It would be of interest to study this example from the toric perspective. The formalism however is much less developed for the cases that require more than one equation.

It is of interest to note that in the expressions (2.2) for the Hodge numbers there occur crossterms that involve both  $\nabla$  and  $\Delta$

$$\begin{aligned}
\bar{\delta} &= \sum_{\text{codim } \tilde{\theta}=2} \text{int}(\theta) \text{int}(\tilde{\theta}) = \sum_{q \in (\dim \tilde{\theta}=3)} (\chi_q - 1) \\
h^{21} &= \sum_{\text{codim } \tilde{\theta}=3} \text{int}(\theta) \text{int}(\tilde{\theta}) = - \sum_{q \in (\dim \tilde{\theta}=2)} (\chi_q - 1) \\
\delta &= \sum_{\text{codim } \tilde{\theta}=4} \text{int}(\theta) \text{int}(\tilde{\theta}) = \sum_{q \in (\dim \tilde{\theta}=1)} (\chi_q - 1)
\end{aligned} \tag{2.5}$$

The quantities  $\delta$  and  $\bar{\delta}$  are respectively the number of non-toric deformations of  $X$  and its mirror, while  $2h^{21}$  is the number of cohomology classes of three cycles. Since the KLR formula(2.4) shows that all  $q$ 's interior to a given  $\tilde{\theta}_q$  have the same arithmetic genus it follows that these crossterms are also equal to  $\pm(\chi_q - 1)$  as given. If  $\dim(\tilde{\theta}) = 4$  then  $\theta_q$  is a vertex and  $\text{int}(\theta_q) = 1$  by convention so

$$\sum_{\text{codim}(\tilde{\theta})=1} (\chi_q - 1) = - \sum_{\text{codim}(\tilde{\theta})=1} \text{int}(\tilde{\theta}_q)$$

which counts the divisors of  $\mathbb{P}_{\nabla}$  that do not intersect the hypersurface. A dual relation obtains also for  $\dim(\tilde{\theta}) = 0$ . We summarize these relations in the following table:

$\dim(\tilde{\theta}_q)$	$\dim(\theta_q)$	$(\chi_q - 1)$ contributes to
4	0	$q$ 's that are not divisors
3	1	$\bar{\delta}$
2	2	$h^{21}$
1	3	$\delta$
0	4	irrelevant monomials

Note that if a divisor has  $\chi \neq 1$  then this divisor contributes to precisely one of the quantities in the third column of the table.



There are also curious relations whose significance we do not understand such as

$$\sum_{q \in \tilde{\theta}} (\chi_q - 1) = \sum_{\tilde{q} \in \theta} (\chi_{\tilde{q}} - 1)$$

that relate  $\chi - 1$  for divisors in the manifold and its mirror. Another such relation is

$$\sum_{q \in \partial \nabla} (\chi_q - 1) = \frac{\chi_E(X)}{6} - (\text{pts}(\nabla) + \text{pts}(\Delta)) + 4$$

or equivalently

$$\sum_{q \in \partial \nabla} \chi_q = \frac{\chi_E(X)}{6} - \text{pts}(\Delta) + 3$$

where  $\partial \nabla$  denotes  $\nabla$  less the interior point and owing to our convention  $\chi_q - 1 = -1$  for  $q$  in a codimension one face of  $\nabla$ .

### 3. Heterotic Structure

#### 3.1. Fibrations

A first statement of the relation between the manifolds that appear in (1.1) may be made in terms of fibrations. A poor but useful notation that we employ is to write  $(\mathcal{F}, \mathcal{B})$  for a manifold that is a fibration over a base  $\mathcal{B}$  with generic fiber  $\mathcal{F}$ . The notation is poor since a manifold is not uniquely specified. There may well be different manifolds that can be realized as fibrations over a given base with the same generic fiber, the difference being due to the manner in which the fibers degenerate over subvarieties of the base. This said, the relation between the manifolds of (1.1) is believed to be the following:

$$\begin{aligned} X = (\mathcal{E}, B^X) &= (K3^Y, B^Z) \quad , \quad Z = (\mathcal{E}, B^Z) \\ B^X &= (\mathbb{P}_1, B^Z) \quad , \quad K3^Y = (\mathcal{E}, \mathbb{P}_1) \end{aligned} \tag{3.1}$$

with  $\mathcal{E}$  denoting an elliptic curve and  $K3^Y$  denoting a  $K3$  manifold. The suffix  $Y$  refers to another Calabi–Yau threefold to which we shall refer as we proceed. In other words,  $X$  is an elliptic fibration over a base  $B^X$ , with  $B^X$  a  $\mathbb{P}_1$ -fibration over a two dimensional base  $B^Z$ . The manifold  $Z$  of the heterotic compactification is then an elliptic fibration over this same two dimensional base. The second representation of  $X$  states that it is also a fibration over  $B^Z$  with fiber an elliptic  $K3$ .

The purpose of the present paper is, in part, to study the relations between these manifolds in the toric context for which some degree of control is afforded by the relation between Calabi–Yau manifolds and reflexive polyhedra and the observation of [15] that the fibration structure of a manifold specified by such a polyhedron is visible in the polyhedron. Although the nature of the bundle  $V^X$  is not properly understood in terms of the toric data, nevertheless some information is available. For example, the structure group  $G^X$  of  $V^X$  can be read off from the polyhedron [16,17]. Another issue that we study in the toric context is that of the existence of divisors that give rise to a superpotential through non-perturbative effects. Proceeding loosely, the integral points,  $q$ , of the dual polyhedron,  $\nabla^X$ , of  $X$  are in direct correspondence with the divisors of  $X$ . Moreover, Klemm *et al.*[12] have formulated a simple and elegant criterion that distinguishes the points corresponding to the divisors of arithmetic genus one. The toric context, though not general, permits a study of examples and a certain systematization which we find useful.

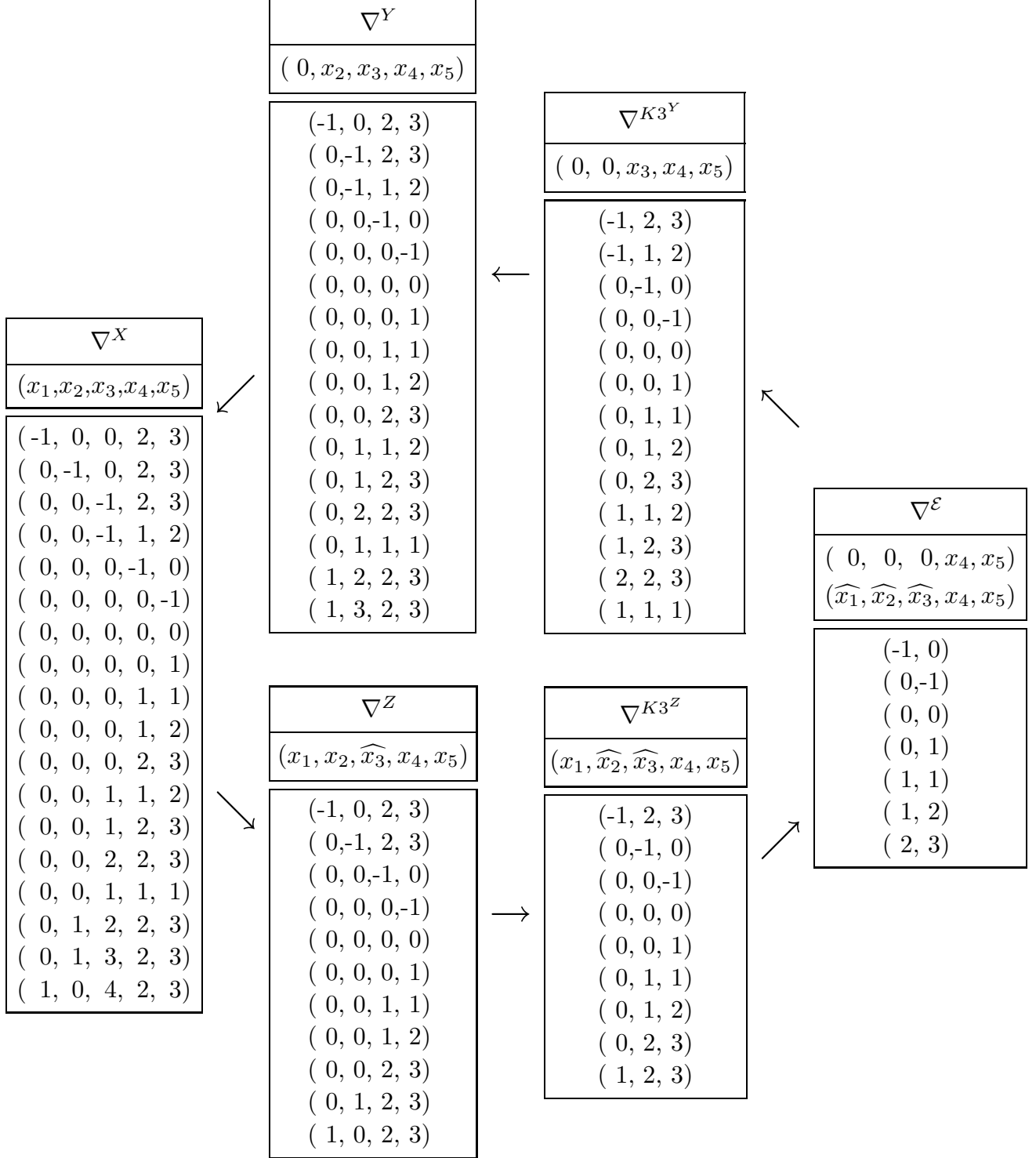
As noted above a reflexive polyhedron corresponding to a Calabi–Yau manifold that is a fibration  $(\mathcal{F}, \mathcal{B})$  has a slice corresponding to the dual polyhedron of the fiber  $\mathcal{F}$  and this enables us to establish a standard coordinate system for the polyhedra. It is perhaps easiest to see this at work in an example. The first column of Table 3.1 lists the integral points of the dual polyhedron of an interesting example which we shall denote by  $X$  throughout this article.

We take the Cartesian coordinates  $(x_1, x_2, x_3, x_4, x_5)$  for the  $\mathbb{R}^5$  in which the polyhedron is embedded. The points that lie in the hyperplane  $\{x_1 = 0\}$  form  $\nabla^Y$ , the dual polyhedron of a Calabi–Yau threefold,  $Y$ . These points are listed in the upper table of column two of Table 3.1. The points with  $\{x_1 = x_2 = 0\}$  form  $\nabla^{K3^Y}$ , the dual polyhedron of a  $K3$  surface associated to  $Y$ . The points with  $\{x_1 = x_2 = x_3 = 0\}$  form the dual polyhedron,  $\nabla^\mathcal{E}$ , of the Weierstrass torus  $\mathcal{E} = \mathbb{P}^{(1,2,3)}[6]$ . Finally the three points with  $\{x_1 = x_2 = x_3 = x_4 = 0\}$  are the dual polyhedron of a zero-dimensional Calabi–Yau manifold<sup>3</sup>. The lower route through the table is realised by making the indicated projections. The coordinates with hats are projected out in this process. Thus  $\nabla^X \rightarrow \nabla^Z$  corresponds to the projection  $(x_1, x_2, x_3, x_4, x_5) \rightarrow (x_1, x_2, x_4, x_5)$ . Note that the fibration manifests additional structure, not only does the dual polyhedron of  $\mathcal{E}$  appear as a slice but also as a projection onto the last two coordinates. This is related to the fact that  $\nabla^\mathcal{E}$  is self dual.

We would like now to discuss the bases of the elliptic Calabi–Yau and  $K3$  fibrations which we denote by  $B^X$ ,  $B^Y$  and  $B^Z$  respectively. We have noted that we can see the dual polyhedron of the fiber,  $\mathcal{E}$ , of the fibration  $X = (\mathcal{E}, B^X)$  as the slice  $\{x_1 = x_2 = x_3 = 0\}$  of  $\nabla^X$ . The base  $B^X$  may also be seen as the *projection* onto the first three coordinates. This gives us the points of the first column of Table 3.2 the rays from the origin through these points yield the *fan* of  $B^X$ . Note also that  $B^X$  can be obtained not only as the projection to the first three coordinates but also as the slice  $\{x_4 = 2, x_5 = 3\}$ , which is a three-face of  $\nabla^X$ . It is one of the observations of [15] that the roles of injections and projections are interchanged by mirror symmetry so the fact that  $\nabla^\mathcal{E}$  and  $B^X$  are visible both as injections and projections has the consequence that the mirror,  $\tilde{X}$ , of  $X$  is also an elliptic fibration  $\tilde{X} = (\mathcal{E}, B^{\tilde{X}})$ , where in this relation  $B^{\tilde{X}}$  denotes the base of the mirror fibration and we write  $\mathcal{E}$  for the fiber in place of  $\tilde{\mathcal{E}}$  since  $\nabla^\mathcal{E}$  is self mirror.  $B^X$  is not Calabi–Yau so there is no notion of a mirror of  $B^X$ . For the Calabi–Yau fibration  $X = (Y, \mathbb{P}_1)$  we have already

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<sup>3</sup> We can think of a zero-dimensional Calabi–Yau manifold as  $\mathbb{P}_1[2]$  with equation  $\xi_1^2 + \xi_1\xi_2 + \xi_2^2 = 0$ . The Newton polyhedron associated with this equation consists of three points in a straight line. After a change of coordinates these become the points  $x = -1, 0, 1$ . This trivial reflexive polyhedron is self-dual



**Table 3.1:** The dual polyhedron for  $X$  with the data of the associated fibrations. The  $\nabla$ 's on the upper level are linked by a series of injections while those on the lower level are related by projections.

noted that  $\nabla^Y$  is the slice  $\{x_1 = 0\}$  of  $\nabla^X$ . The fan of the  $\mathbb{P}_1$  consists of the three points  $x_1 = -1, 0, 1$  obtained by projecting  $\nabla^X$  onto the first coordinate. The threefold  $Y$  is itself an elliptic fibration,  $Y = (\mathcal{E}, B^Y)$ , over a base  $B^Y$  whose toric data are obtained by projecting  $\nabla^Y$  to its first two coordinates. This gives the upper table of the second column of Table 3.2. As for the  $K3$ -fibration  $X = (K3^Y, B^Z)$ , we have seen the  $K3^Y$  as the slice  $\{x_1 = x_2 = 0\}$  and we see  $B^Z$  as the projection onto the first two coordinates, the result being the lower table of the second column of Table 3.2. The base of the  $K3$ -fibration is in fact the base of the fibration  $Z = (\mathcal{E}, B^Z)$  as we shall see. Note however that  $Y$  and the  $K3$  are not projections onto any slices so we might not expect the mirror of  $X$  to be a Calabi–Yau and  $K3$ -fibration, although we shall see presently that it is.

	<table><tr><th><math>B^Y</math></th></tr><tr><td><math>(\ 0, x_2, x_3, \widehat{x_4}, \widehat{x_5})</math></td></tr><tr><td><math>(\ 0, x_2, x_3, \ 2, \ 3)</math></td></tr></table>	$B^Y$	$(\ 0, x_2, x_3, \widehat{x_4}, \widehat{x_5})$	$(\ 0, x_2, x_3, \ 2, \ 3)$													
$B^Y$																	
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	<table><tr><td><math>(-1, \ 0)</math></td></tr><tr><td><math>(\ 0, -1)</math></td></tr><tr><td><math>(\ 0, \ 0)</math></td></tr><tr><td><math>(\ 0, \ 1)</math></td></tr><tr><td><math>(\ 0, \ 2)</math></td></tr><tr><td><math>(\ 1, \ 2)</math></td></tr><tr><td><math>(\ 1, \ 3)</math></td></tr></table>	$(-1, \ 0)$	$(\ 0, -1)$	$(\ 0, \ 0)$	$(\ 0, \ 1)$	$(\ 0, \ 2)$	$(\ 1, \ 2)$	$(\ 1, \ 3)$	<table><tr><th><math>\mathbb{P}_1^Z</math></th></tr><tr><td><math>(0 \ , x_2, \widehat{x_3}, \widehat{x_4}, \widehat{x_5})</math></td></tr><tr><td><math>(0 \ , x_2, \widehat{x_3}, \ 2, \ 3)</math></td></tr></table>	$\mathbb{P}_1^Z$	$(0 \ , x_2, \widehat{x_3}, \widehat{x_4}, \widehat{x_5})$	$(0 \ , x_2, \widehat{x_3}, \ 2, \ 3)$					
$(-1, \ 0)$																	
$(\ 0, -1)$																	
$(\ 0, \ 0)$																	
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$(\ 0, \ 2)$																	
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<table><tr><th><math>B^X</math></th></tr><tr><td><math>(x_1, x_2, x_3, \widehat{x_4}, \widehat{x_5})</math></td></tr><tr><td><math>(x_1, x_2, x_3, \ 2, \ 3)</math></td></tr></table>	$B^X$	$(x_1, x_2, x_3, \widehat{x_4}, \widehat{x_5})$	$(x_1, x_2, x_3, \ 2, \ 3)$	<table><tr><td><math>(-1, \ 0, \ 0)</math></td></tr><tr><td><math>(\ 0, -1, \ 0)</math></td></tr><tr><td><math>(\ 0, \ 0, -1)</math></td></tr><tr><td><math>(\ 0, \ 0, \ 0)</math></td></tr><tr><td><math>(\ 0, \ 0, \ 1)</math></td></tr><tr><td><math>(\ 0, \ 0, \ 2)</math></td></tr><tr><td><math>(\ 0, \ 1, \ 2)</math></td></tr><tr><td><math>(\ 0, \ 1, \ 3)</math></td></tr><tr><td><math>(\ 1, \ 0, \ 4)</math></td></tr></table>	$(-1, \ 0, \ 0)$	$(\ 0, -1, \ 0)$	$(\ 0, \ 0, -1)$	$(\ 0, \ 0, \ 0)$	$(\ 0, \ 0, \ 1)$	$(\ 0, \ 0, \ 2)$	$(\ 0, \ 1, \ 2)$	$(\ 0, \ 1, \ 3)$	$(\ 1, \ 0, \ 4)$	<table><tr><td><math>(-1)</math></td></tr><tr><td><math>(\ 0)</math></td></tr><tr><td><math>(\ 1)</math></td></tr></table>	$(-1)$	$(\ 0)$	$(\ 1)$
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	<table><tr><th><math>B^Z</math></th></tr><tr><td><math>(x_1, x_2, \widehat{x_3}, \widehat{x_4}, \widehat{x_5})</math></td></tr><tr><td><math>(x_1, x_2, \widehat{x_3}, \ 2, \ 3)</math></td></tr></table>	$B^Z$	$(x_1, x_2, \widehat{x_3}, \widehat{x_4}, \widehat{x_5})$	$(x_1, x_2, \widehat{x_3}, \ 2, \ 3)$	<table><tr><th><math>\mathbb{P}_1^Y</math></th></tr><tr><td><math>(x_1, \widehat{x_2}, \widehat{x_3}, \widehat{x_4}, \widehat{x_5})</math></td></tr><tr><td><math>(x_1, \widehat{x_2}, \widehat{x_3}, \ 2, \ 3)</math></td></tr></table>	$\mathbb{P}_1^Y$	$(x_1, \widehat{x_2}, \widehat{x_3}, \widehat{x_4}, \widehat{x_5})$	$(x_1, \widehat{x_2}, \widehat{x_3}, \ 2, \ 3)$									
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	<table><tr><td><math>(-1, \ 0)</math></td></tr><tr><td><math>(\ 0, -1)</math></td></tr><tr><td><math>(\ 0, \ 0)</math></td></tr><tr><td><math>(\ 0, \ 1)</math></td></tr><tr><td><math>(\ 1, \ 0)</math></td></tr></table>	$(-1, \ 0)$	$(\ 0, -1)$	$(\ 0, \ 0)$	$(\ 0, \ 1)$	$(\ 1, \ 0)$	<table><tr><td><math>(-1)</math></td></tr><tr><td><math>(\ 0)</math></td></tr><tr><td><math>(\ 1)</math></td></tr></table>	$(-1)$	$(\ 0)$	$(\ 1)$							
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**Table 3.2:** The points corresponding to the bases of the fibrations.

To summarize thus far, the coordinates of  $\nabla^X$  relate  $B^X$ ,  $B^Y$ ,  $B^Z$  and  $\mathcal{E}$  as follows

$$\vec{x} = \underbrace{(x_1, x_2, x_3)}_{B^Z} \underbrace{(x_4, x_5)}_{B^Y} \quad \begin{matrix} B^X & \mathcal{E} \\ \underbrace{\hspace{1cm}} & \underbrace{\hspace{1cm}} \end{matrix}$$

The reader wishing to acquire dexterity with seeing the various fibrations should check from Table 3.2 that

$$B^X = (\mathbb{P}_1, B^Z) \quad \text{and} \quad B^X = (B^Y, \mathbb{P}_1) .$$

We want to find  $Z = (\mathcal{E}, B^Z)$  in  $\nabla^X$  so we need to project out the coordinate  $x_3$ . Thus  $\nabla^Z$  can be realized by performing the projection  $(x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, x_2, x_4, x_5)$ . This yields the sixth column of Table 3.1. Another way of stating this is that  $Z$  is being identified as the base of a fibration. This statement seems to involve mirror symmetry in a nontrivial way. The fibrations being

$$\begin{aligned} X &= (Y, \mathbb{P}_1) = (\mathbb{Z}_2, \mathbb{P}(\nabla^Z)) \\ \tilde{X} &= (\tilde{Z}, \mathbb{P}_1) = (\mathbb{Z}_2, \mathbb{P}(\Delta^Y)) \end{aligned}$$

where  $\mathbb{P}(\nabla^Z)$  and  $\mathbb{P}(\Delta^Y)$  denote the toric manifolds corresponding to the fans over the faces of  $\nabla^Z$  and  $\Delta^Y$  respectively.

A class of fourfolds  $X$  with the structure  $X = (\mathcal{E}, B^X)$  with  $B^X = (\mathbb{P}_1, B^Z)$  has been discussed in [12]. The idea is to take  $B^X$  of the form  $(\mathbb{P}_1, (\mathbb{P}_1, \mathbb{P}_1))$ . The inner fibration  $(\mathbb{P}_1, \mathbb{P}_1)$  is taken to be a fiber bundle, the Hirzebruch surface<sup>4</sup>  $\mathbb{F}_m$ . The wrapping of the outer  $\mathbb{P}_1$  is specified by two further integers corresponding to the wrapping of this  $\mathbb{P}_1$  about each of the other two. The resulting manifold is denoted by  $\mathbb{F}_{mnp}$  in [12]. For many values of these integers, it is possible to define in terms of toric data manifolds  $X = (\mathcal{E}, \mathbb{F}_{mnp})$  in such a way that  $X$  is Calabi–Yau. By this we mean here that we can associate a Newton polyhedron with  $X$  and this polyhedron has the property of being reflexive.

These reflexive polyhedra have interesting structure corresponding to the fact that in addition to the structure (1.1) we have also

$$X = (Y, \mathbb{P}_1) \quad , \quad Y = (K3^Y, \mathbb{P}_1) \quad , \quad K3^Y = (\mathcal{E}, \mathbb{P}_1) \quad , \quad \mathcal{E} = (\mathbb{Z}_2, \mathbb{P}_1) \quad , \quad (3.2)$$

---

<sup>4</sup> Notice that this already shows how loose the notation  $(\mathbb{P}_1, \mathbb{P}_1)$  is since even for the case of a fiber bundle there is one of these for each integer  $m$ .

with  $Y$  a Calabi–Yau threefold. Now, this structure manifests itself in the dual polyhedron  $\nabla^X$  of  $X$  in a very simple way:  $\nabla^X$  contains a codimension one slice which is  $\nabla^Y$ , the dual polyhedron of  $Y$  and this structure is repeated;  $\nabla^Y$  contains  $\nabla^{K3^Y}$ , the dual polyhedron of the  $K3$  surface as a slice and  $\nabla^{K3^Y}$  contains  $\nabla^\mathcal{E}$  the triangle of the Weierstrass polynomial as a slice. Finally,  $\nabla^\mathcal{E}$  contains a line with three points corresponding to a zero dimensional Calabi–Yau manifold. In other words we have a series of injections

$$\nabla^X \longleftarrow \nabla^Y \longleftarrow \nabla^{K3^Y} \longleftarrow \nabla^\mathcal{E} \longleftarrow \nabla^{\mathbb{Z}_2} . \quad (3.3)$$

A point that is important is that this structure imposes a natural coordinate system on the polyhedron.  $\nabla^X$  is five-dimensional; within it is a four-plane containing  $\nabla^Y$ ; within this a three-plane containing  $\nabla^{K3}$ ; within this a two-plane containing  $\nabla^\mathcal{E}$ ; and finally within this a line corresponding to the zero dimensional Calabi–Yau  $\nabla^{\mathbb{Z}_2}$ .

There is a further important property of this class of manifolds which is that there is also a hierarchy of *projections* that relate  $X$  to  $Z$ :

$$\nabla^X \longrightarrow \nabla^Z \longrightarrow \nabla^{K3^Z} \longrightarrow \nabla^\mathcal{E} \longrightarrow \nabla^{\mathbb{Z}_2} \quad (3.4)$$

It is one of the observations of [15] that the roles of injections and projections are interchanged by mirror symmetry so the mirror of each such  $X$  has also a structure analogous to (3.3) and (3.4) with the replacements  $Y \rightarrow \tilde{Z}$  and  $Z \rightarrow \tilde{Y}$ . Here and in the following tildes are used to denote the mirror of a given manifold.

In these notes we shall be primarily concerned with the KLRY spaces and their mirrors (these spaces will be discussed in details in §4). The polyhedra for these classes are very different, the KLRY spaces have dual polyhedra that are small, typically with 20–70 points, while their Newton polyhedra (which are the dual polyhedra of the mirrors) are large with typically 20,000–70,000 points. The structure of these fourfolds suggests a precise specification of the threefold  $Z^X$ , given  $X$ , by defining  $Z$  in terms of its mirror

$$F[X] = \text{Het}[Z, V^X] \quad , \quad \tilde{X} = (\tilde{Z}, \mathbb{P}_1) \quad , \quad (3.5)$$

where in the second relation  $\tilde{X}$  and  $\tilde{Z}$  are the mirrors of  $X$  and  $Z$  respectively. As we show in §2, the relation  $\tilde{X} = (\tilde{Z}, \mathbb{P}_1)$ , and hence  $Z$ , can be given a precise meaning in virtue of the natural projections mentioned above. For  $X$  the KLRY space  $X_{mnp}$ , this relation gives what we would expect  $Z = Z_m = (\mathcal{E}, \mathbb{F}_m)$ , the elliptic fibration of the Hirzebruch surface that is familiar from [18]. However, for  $X$  the mirror of a KLRY space, the fact that we obtain a sensible definition of  $Z$  in this way is far from trivial.

The occurrence of divisors that lead to superpotentials turns out to be a rather involved subject. A first observation that is perhaps counterintuitive given [5,14] is that divisors of the fourfold with arithmetic genus one are ubiquitous at least in the class of fourfolds that we study. (Of more than 3000 manifolds all have these divisors and all their mirrors have them also.) What is less clear is how to deal with compactifications to four dimensions for which we are interested in divisors of arithmetic genus one that are of the form  $\pi^{-1}(R)$  where  $\pi$  denotes the projections  $\pi : X \longrightarrow B^X$  onto the base of the fibration and  $R$  denotes a divisor of  $B^X$ . Here the situation is clearest when  $\pi^{-1}(R)$  consists of a single component which is a divisor of  $X$  of arithmetic genus one. Frequently however, the preimage  $\pi^{-1}(R)$  consists of several irreducible components with non-trivial intersection. These preimages are precisely the ones that give rise to the gauge group  $G^X$  of the heterotic model.



## 4. Construction of $X$

### 4.1. The KLRV Spaces

The KLRV spaces,  $X_{mnp}$ , provide many examples of elliptically fibered fourfolds with the structure (3.1). These spaces are defined by toric data

$$Q_{mnp} = \begin{pmatrix} 1 & 1 & m & 0 & p & 0 & 2(m+p+2) & 3(m+p+2) & 0 \\ 0 & 0 & 1 & 1 & n & 0 & 2(m+2) & 3(m+2) & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 \end{pmatrix} \quad (4.1)$$

with the three integers having the ranges  $1 \leq m, n \leq 12$  and  $0 \leq p \leq \frac{1}{2}n(m+2)$ . What is meant by this is, of course, that we have 9 coordinates  $(\xi_1, \xi_2, \dots, \xi_9)$  that are identified under 4 scaling symmetries with weights given by the rows of  $Q$ . Now the allowed monomials (those that have the same multidegree as the fundamental monomial  $\xi_1 \xi_2 \cdots \xi_9$ ) are points in a 9-dimensional space. However in virtue of the scaling relations these points lie in a 5-dimensional plane. Thus the monomials associated with (4.1) give rise to a 5-dimensional Newton polyhedron,  $\Delta$ . This polyhedron is reflexive and so corresponds to a Calabi–Yau fourfold.

Perhaps the simplest way to motivate the particular fourfold that we study is to describe first a simple but inconsistent model associated with the manifold  $X_{234}$  which we denote by  $\tilde{X}$  to save writing. The model is inconsistent for our purposes since, despite the fact that it appears to be an elliptic fibration by construction nevertheless, as we shall see, it fails to be an elliptic fibration. We shall discuss this carefully in the following however the point at issue is whether the embedding space  $\mathbb{P}^\nabla$  is an elliptic fibration. Such a fibration is expressed torically if every cone in the fan of  $\tilde{X}$  projects to some cone of the fan of the base. This is what fails for  $\tilde{X}$ . Repairing the fibration leads to a more complicated but viable model.

To begin however consider then the dual polyhedron  $\tilde{\nabla}$  corresponding to  $\tilde{X}$  shown below. The divisors  $E_1$  and  $\tilde{E}$  are associated with an  $SU(2)$  gauge group. The fact that the groups are as seen from the polyhedron can be verified by performing a calculation in the Cox coordinate ring. In our case, this has been checked by A. Klemm. As we have already stressed, the fibration  $\tilde{X} \longrightarrow B$  is important for us and it is related to the

projection onto the first three coordinates of the points of  $\tilde{\nabla}$ .

Vertex	$\tilde{\nabla}$	Divisor
$V_1$	$(-1, 0, 0, 2, 3)$	
$V_2$	$(0, -1, 0, 2, 3)$	
$V_3$	$(0, 0, -1, 2, 3)$	
$V_4$	$(0, 0, 0, -1, 0)$	
$V_5$	$(0, 0, 0, 0, -1)$	
$\frac{1}{2}(V_3 + E_1)$	$(0, 0, 0, 2, 3)$	$B$
$\frac{1}{3}(V_2 + V_4 + V_7)$	$(0, 0, 1, 1, 2)$	$\tilde{E}$
$\frac{1}{2}(V_2 + F)$	$(0, 0, 1, 2, 3)$	$E_1$
$\frac{1}{2}(V_1 + V_6)$	$(0, 1, 2, 2, 3)$	$F$
$V_7$	$(0, 1, 3, 2, 3)$	$G$
$V_6$	$(1, 2, 4, 2, 3)$	$Y$

#### 4.2. Constructing the Fan for $\tilde{X}$

The polyhedron  $\tilde{\nabla}$  is not a simplex. However, we can start by taking the cones over its faces

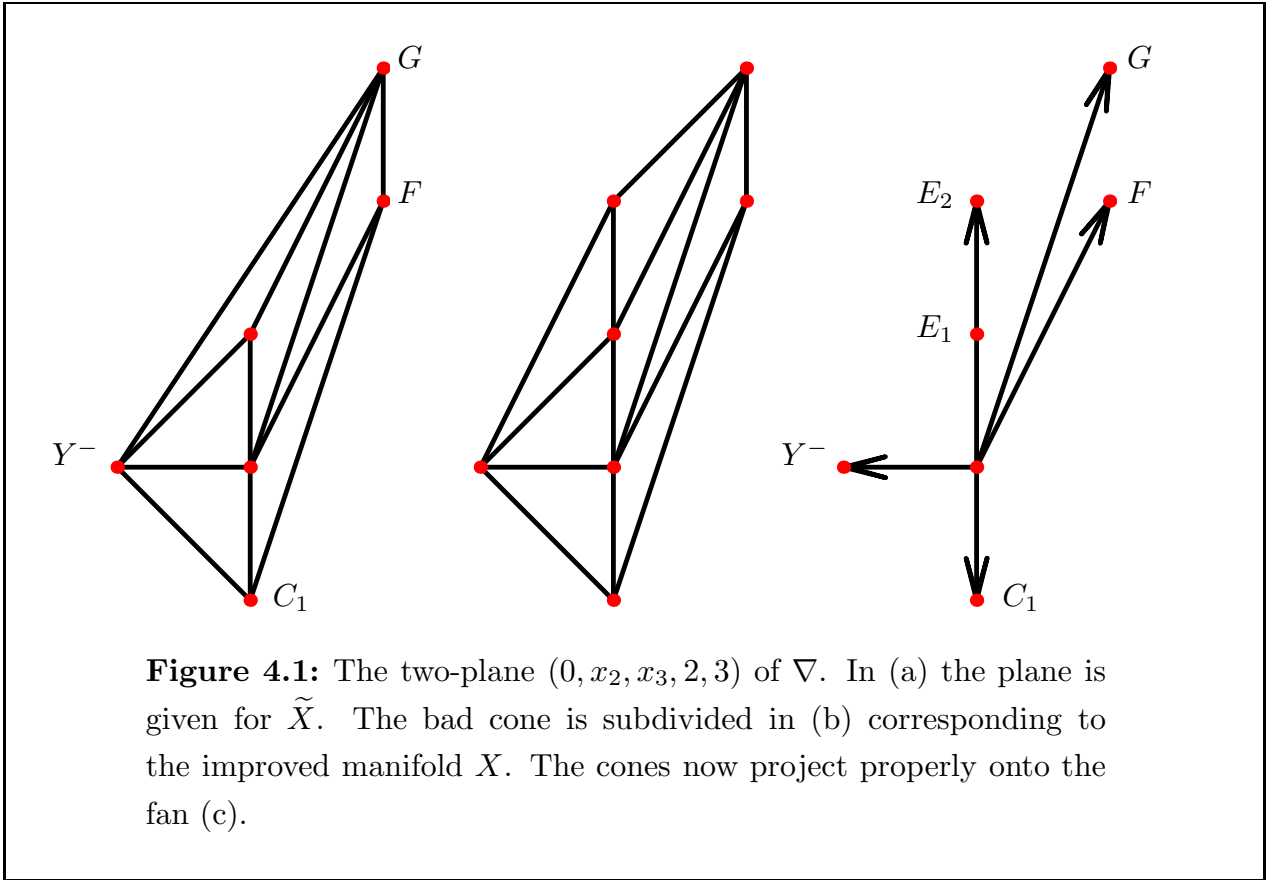
$$\{V_2V_4V_5V_6V_7, V_2V_3V_4V_5V_6, V_1V_2V_3V_4V_5, V_1V_2V_3V_4V_6V_7, \\ V_1V_2V_3V_5V_6V_7, V_1V_3V_4V_5V_6, V_1V_2V_4V_5V_7, V_1V_4V_5V_6V_7\}$$

Two of these cones  $V_1V_2V_3V_4V_6V_7$  and  $V_1V_2V_3V_5V_6V_7$  are not simplicial. They correspond to facets of  $\tilde{\nabla}$  that share a common 3-face  $V_1V_2V_3V_6V_7$ . We can see how to perform a triangulation by drawing the  $V_2V_3V_7F$ -plane as well as the  $V_2V_4V_7$  2-face. The following rules effect the triangulation:

$$\begin{aligned} V_1V_6 &\longrightarrow \{V_1F, FV_6\} \\ V_2V_3V_7F &\longrightarrow \{V_2V_3B, V_2BE_1, V_2E_1V_7, BE_1V_7, BFV_7, V_3BF\} \\ V_2V_4V_7 &\longrightarrow \{V_2\tilde{E}V_4, V_4\tilde{E}V_7, V_7\tilde{E}V_2\} \end{aligned}$$

It is necessary to check that the  $\tilde{X}$  is actually an elliptic fibration, that is, that the map  $\pi : \tilde{X} \longrightarrow B$  is smooth. We may ensure this by requiring that each cone of the fan for

$\tilde{X}$  should project onto some cone of the fan for the base  $B$ . As we will show now, this is not the case for  $\tilde{\nabla}$ . The problem arises in connection with the triangulation of the three face  $(x_1, x_2, x_3, 2, 3)$  corresponding to  $\Sigma^B$ . In Figure 4.1, we draw the two plane  $(0, x_2, x_3, 2, 3,)$  that lies within this face. The cones of  $\Sigma^{\tilde{X}}$  intersect this plane in the regions indicated. The problem comes from the cone from the origin of  $\tilde{\nabla}$  (which is out of the plane) which intersects this plane in the triangle  $V_2 E_1 G$ . If we project this cone onto the plane clearly it projects onto the union of the cone generated by  $V_1$  and  $E_1$  and the cone generated by  $E_1$  and  $G$ . The simplest way to fix the problem is to add the point  $E_2 \sim (0, 0, 2, 2, 3)$  to  $\tilde{\nabla}$  as in Figure 4.1b. This divides the troublesome cone in two so that the fan for  $\tilde{X}$  now projects nicely.



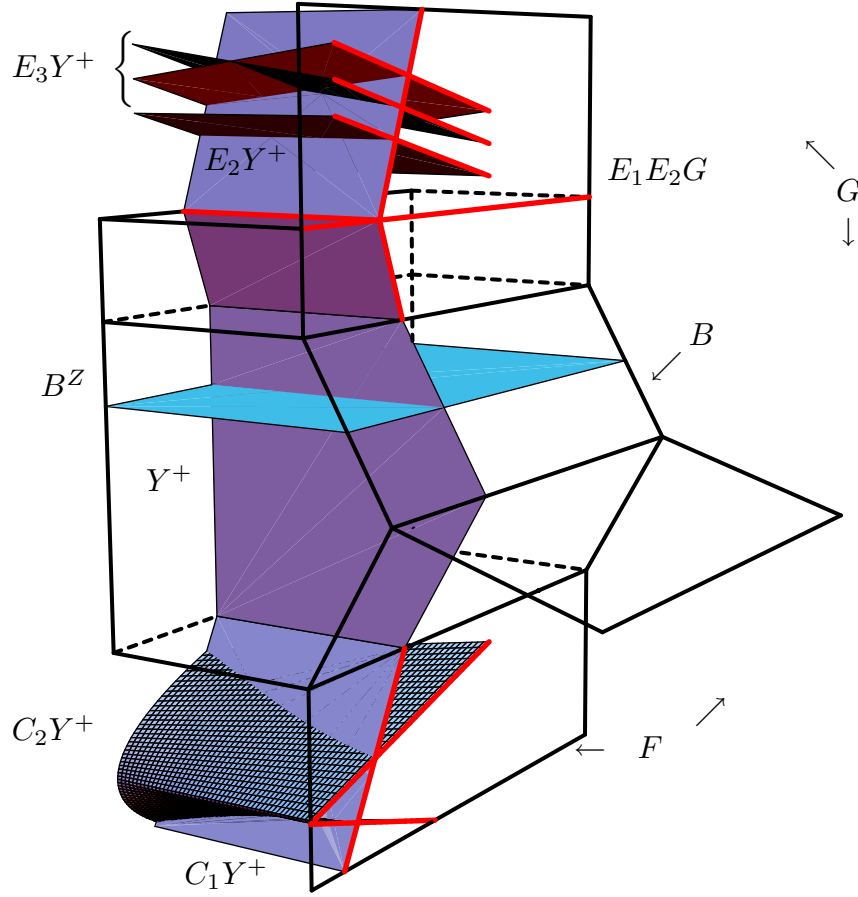
When we take the convex hull of  $\tilde{\nabla} \cup \{E_2\}$  we find that the new polyhedron contains also the point  $E_3 \sim (0, 0, 1, 1, 1)$ . Moreover, the point  $\tilde{E}$  now lies in the interior of a codimension one face. In this way, the  $SU(2)$  associated with  $\{E_1, \tilde{E}\}$  is replaced by a  $G_2$  associated with  $\{E_1, E_2, E_3\}$ .

At the level of practical calculation, note that the preimage of the ray  $(0, 0, 1)$  of the base of  $\tilde{X}$  is the divisor  $E_1 + \tilde{E}$ . It follows that  $(E_1 + \tilde{E})^4$  should vanish since the intersection calculation pulls back from the intersection calculation on the base. This consistency check fails for the fan for  $\tilde{X}$ .

We are not yet quite done with the changes. In order to enforce this condition, we have to add the point  $C_2 \sim (1, 0, -1, 1, 2)$ . The data for our consistent fourfold is displayed in Table 4.1. Note that  $\{C_1, C_2\}$  correspond to an additional  $SU(2)$  gauge group. We have added the divisor  $C_2$ , which could have been omitted, in order to show how we may build up the group. We make some further comments about how to build up the groups in §4.3 below. Table 4.1 summarizes the polyhedron and divisors for  $X$ .

Relation to vertices	$\chi$	$\nabla^X$	Divisor
$V_1$	0	$(-1, 0, 0, 2, 3)$	$Y^+$
$V_2$	0	$(0, -1, 0, 2, 3)$	$Y^- = F + G$
$V_3$	$1^*$	$(0, 0, -1, 2, 3)$	$C_1$
$V_9$	$1^*$	$(0, 0, -1, 1, 2)$	$C_2$
$V_4$	-89	$(0, 0, 0, -1, 0)$	$2H + E_3 - C_2$
$V_5$	-368	$(0, 0, 0, 0, -1)$	$3H + E_3 - C_2$
$\frac{1}{2}(V_3 + E_1)$	$1^*$	$(0, 0, 0, 2, 3)$	$B$
$\frac{1}{2}(V_3 + 2E_2)$	$1^*$	$(0, 0, 1, 2, 3)$	$E_1$
$\frac{1}{2}(V_1 + V_6)$	$1^*$	$(0, 0, 2, 2, 3)$	$E_2$
$\frac{1}{2}(V_5 + E_2)$	0	$(0, 0, 1, 1, 1)$	$E_3 = C_1 + C_2 - (E_1 + 2E_2 + 2F + 3G + 4Y^+)$
$V_7$	$1^*$	$(0, 1, 2, 2, 3)$	$F$
$V_8$	$1^*$	$(0, 1, 3, 2, 3)$	$G$
$V_6$	0	$(1, 0, 4, 2, 3)$	$Y^+$
$h^{11} = 8, \quad h^{31} = 2897, \quad h^{21} = 1, \quad h^{22} = 11662, \quad \chi_E = 17472$			
$H = B + C_1 + C_2 + E_1 + E_2 + 2F + 2G + 2Y^+$			

**Table 4.1:** The divisors for the manifold  $X$ .



**Figure 4.2:** A sketch of how the various divisors intersect the base  $B$  of the elliptic fibration showing also the degenerate fibers corresponding to the groups  $SU_2$  and  $G_2$ . The surfaces shown as  $E_3Y^+$  are really a single connected surface. The ‘components’ that are shown meet in pairs. The surface  $C_2Y^+$  is ruled by quadrics which degenerate, exceptionally, into a pair of lines. This explains the  $1/2$  that appears in the relation  $\ell^3 = \frac{1}{2}C_2FY^+$ .

From [12] we also see that

$$h^{k,0}(E_1) = h^{k,0}(E_2) = h^{k,0}(C_j) = h^{k,0}(F) = h^{0,0}(G) = 0 \text{ for } k = 1, 2, 3$$

and  $h^{0,0}(E_i) = h^{0,0}(C_j) = h^{0,0}(F) = h^{0,0}(G) = 1$ , while  $h^{2,0}(E_3) = 0$  and  $h^{1,0}(E_3) = 1$ .

For our model  $X$  our construction of the fan proceeds similarly to the case of  $\tilde{X}$ . We begin with the cones over the facets of the polyhedron that is the convex hull of the vertices  $V_1, V_2, \dots, V_8$ . This yields the cones

$$\{V_4V_5V_6V_7V_8, V_2V_3V_4V_5V_6, V_1V_3V_4V_5V_6, V_1V_2V_3V_4V_5, V_1V_2V_3V_4V_6V_7V_8, \\ V_1V_2V_3V_5V_6V_7V_8, V_1V_4V_5V_6V_7, V_1V_2V_4V_5V_8, V_1V_4V_5V_7V_8, V_2V_4V_5V_6V_8\}$$

There are two facets which are not simplices,  $V_1V_2V_3V_4V_6V_7V_8$  and  $V_1V_2V_3V_5V_6V_7V_8$ , and these facets have a common 3-face  $V_1V_2V_3V_6V_7V_8 = V_1V_6Y^-C_1FG$  which corresponds to the fan for the base  $B$  of the elliptic fibration (see Table 4.1). To see how to perform a triangulation we make reference to Figure 4.1 which depicts the two-plane  $V_1V_2V_3V_6V_7V_8$  that lies within this three-face. By associating divisors to the points of the figure and noting that  $V_1$  and  $V_6$  are the only points of the three plane that do not lie in the two plane and that these points are joined by a line that passes through  $F$ . We see that the following rules effect the triangulation:

$$\begin{aligned} V_1V_6 &\longrightarrow \{V_1F, FV_6\} \\ V_2V_3V_7V_8F &\longrightarrow \{V_2V_3B, V_2BE_1, V_2E_1V_8, E_1V_7V_8, BE_1V_7, BFV_7, V_3BF\} . \\ V_5V_8 &\longrightarrow \{V_5E_3, E_3V_8\} \end{aligned}$$

This yields a simplicial fan. Finally, we insert  $V_9$  which lies in the cone  $V_3V_4V_5$  by means of the rule:

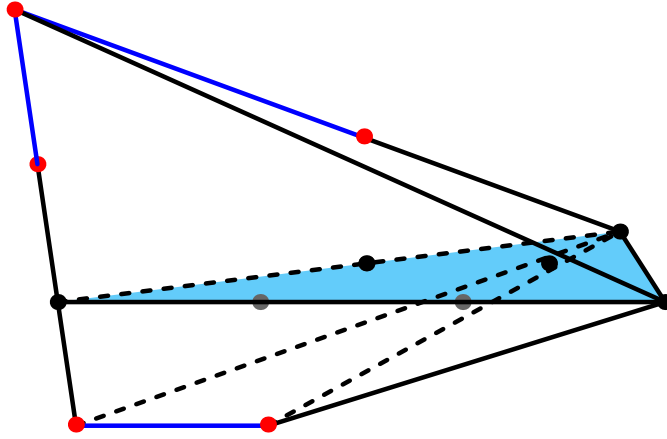
$$V_3V_4V_5 \longrightarrow \{V_3V_4V_9, V_4V_5V_9, V_3V_5V_9\} .$$

The result is a fan of 54 cones  $\Sigma^X = V_1\Sigma^Y \cup V_6\Sigma^Y$ , where  $\Sigma^Y$  denotes the fan of 27 cones  $\{BE_1V_2V_4, BFC_1V_4, BV_2C_1V_4, E_3E_1V_2V_5, BE_1V_2V_5, BFC_1V_5, BV_2C_1V_5, E_3V_2V_4V_5, \\ BFV_4G, BE_1V_4G, BFV_5G, E_3E_1V_5G, BE_1V_5G, E_3V_4V_5G, FV_4V_5G, E_3E_1V_2E_2, \\ E_3V_2V_4E_2, E_1V_2V_4E_2, E_3E_1GE_2, E_3V_4GE_2, E_1V_4GE_2, FC_1V_4C_2, V_2C_1V_4C_2, \\ FC_1V_5C_2, V_2C_1V_5C_2, FV_4V_5C_2, V_2V_4V_5C_2\}$

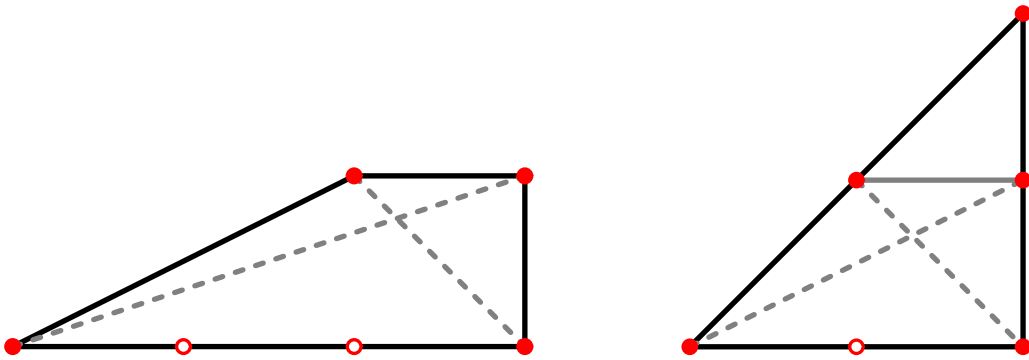
and  $V_1\Sigma^Y$  denotes the set obtained by appending  $V_1$  to each cone of  $\Sigma^Y$  (and similarly for  $V_6\Sigma^Y$ ).

We can in fact express the combinatorics of the fan in a better way. To do this we write each cone as a product and a fan as a sum of cones. Thus with this understanding  $\Sigma^Y = BE_1V_2V_4 + BFC_1V_4 + \dots + V_2V_4V_5C_2$  and  $\Sigma^X = (V_1 + V_6)\Sigma^Y$ . Now with this notation we may express  $\Sigma^Y$  in the form

$$\Sigma^Y = (F + V_2)\Sigma^C + (G + V_2)\Sigma^E + FG\Sigma^E$$



**Figure 4.3:** The polyhedron for  $K^Y$ . The horizontal triangle  $\nabla^\varepsilon$  divides the polyhedron into a top and a bottom. The extended Dynkin diagrams corresponding to the groups  $G_2$  and  $SU(2)$  are seen as the blue lines.



**Figure 4.4:** The two faces of  $\nabla^{K^Y}$  that admit more than one triangulation.

where  $\Sigma^C$  and  $\Sigma^E$  denote the fans over the faces of the half-polyhedra corresponding to the  $SU(2)$  and  $G_2$ . and  $\Sigma^\mathcal{E}$  denotes the fan over the faces of  $\nabla^\mathcal{E}$ .

$$\begin{aligned}\Sigma^C &= BC_1V_4 + C_1C_2V_4 + BC_1V_5 + C_1C_2V_5 + C_2V_4V_5 \\ \Sigma^E &= E_1E_2E_3 + BE_1V_4 + E_1E_2V_4 + E_2E_3V_4 + BE_1V_5 + E_1E_3V_5 + E_3V_4V_5 \\ \Sigma^\mathcal{E} &= BV_4 + BV_5 + V_4V_5 .\end{aligned}$$

Could we have chosen different fans? The answer is yes however if we restrict to fans that project to the fan for  $B^X$  then there are just four choices and these are related by flops corresponding to the fact that two of the faces of  $\nabla^{K^Y}$  each admit two different triangulations as in Figure 4.4. The yukawa couplings for these four fans are however the same which indicates that the flops affect the embedding space but not the hypersurface  $X$ .

#### 4.3. Extending the Groups

It is of course also possible to extend the gauge group by adding points to  $\nabla$ . We think of this as building up the half-polyhedra that project down onto the divisors of  $B$  and wish to show how this may be implemented on the fan. Suppose we begin by seeking to build up a half-polyhedron over the divisor  $F$ . We change notation by denoting  $F$  by  $F_1$ , so that the new points are  $F_2, F_3$  etc. and by denoting  $G$  by  $G_1$  since we will want also to add a group over  $G$ . Our starting point is the fan

$$\Sigma^Y = (F_1 + V_2)\Sigma^{\text{bot}} + (G_1 + V_2)\Sigma^{\text{top}} + F_1G_1\Sigma^\mathcal{E}$$

One checks that the new points  $F_2, F_3, \dots$  all lie in the cone  $F_1V_4V_5$  of  $\Sigma^Y$ . This cone occurs as a face in two of the terms in  $\Sigma^Y$ . It occurs both in  $F_1\Sigma^\mathcal{E}$  and  $F_1\Sigma^C$  since both  $\Sigma^C$  and  $\Sigma^\mathcal{E}$  contain the cone  $V_4V_5$ . Let  $\Sigma^{F_1}$  denote the trivial half-polyhedron over  $F_1$ , that is the half-polyhedron with no extra points, and denote by  $\Sigma^F$  the half-polyhedron corresponding to the new group. We will write  $\Sigma^{F_1} \longrightarrow \Sigma^F$  for the process of extending the fan to the fan over the faces of the new half-polyhedron.

For the term  $F_1\Sigma^\mathcal{E}$  we observe that

$$F_1\Sigma^\mathcal{E} = \Sigma^{F_1} \longrightarrow \Sigma^F .$$

While for the term  $F_1V_4V_5$  in  $F_1\Sigma^C$  we note that

$$F_1V_4V_5 = \Sigma^{F_1} - F_1B(V_4 + V_5)$$



so we have

$$\begin{aligned}
F_1 \Sigma^C &= F_1 (\Sigma^C - C_2 V_4 V_5) + C_2 (F_1 V_4 V_5) \\
&= F_1 (\Sigma^C - C_2 V_4 V_5) + C_2 (\Sigma^{F_1} - F_1 B(V_4 + V_5)) \\
&\longrightarrow F_1 (\Sigma^C - C_2 V_4 V_5) + C_2 (\Sigma^F - F_1 B(V_4 + V_5)) .
\end{aligned}$$

In this way we see that  $\Sigma^Y \longrightarrow \tilde{\Sigma}^Y$  with

$$\tilde{\Sigma}^Y = (F_1 + V_2) \Sigma^C + (G_1 + V_2) \Sigma^E + (G_1 + C_2) \Sigma^F - F_1 C_2 \Sigma^E .$$

Note that the function of the last term is to remove terms that are present in  $F_1 \Sigma^C$  so there are really no minus signs in this expression.

Now let us add a group over  $G_1$ . The new points all lie in the cone  $G_1 V_4 V_5$  which is contained in the terms  $G_1 \Sigma^F$  and  $G_1 \Sigma^E$ . In  $\Sigma^F$  there is a single cone of the form  $F_j V_4 V_5$  and we denote this divisor  $F_j$  by  $F_{\max}$ . For  $G_1 \Sigma^F$  we write

$$\begin{aligned}
G_1 \Sigma^F &= G_1 (\Sigma^F - F_{\max} V_4 V_5) + F_{\max} (\Sigma^{G_1} - G_1 B(V_4 + V_5)) \\
&\longrightarrow G_1 (\Sigma^F - F_{\max} V_4 V_5) + F_{\max} (\Sigma^G - G_1 B(V_4 + V_5)) \\
&= G_1 \Sigma^F + F_{\max} \Sigma^G - F_{\max} G_1 \Sigma^E .
\end{aligned}$$

Similarly

$$G_1 \Sigma^E \longrightarrow G_1 \Sigma^E + E_3 \Sigma^G - E_3 G_1 \Sigma^E .$$

In this way we arrive at a fan

$$\begin{aligned}
\tilde{\tilde{\Sigma}} &= (F_1 + V_2) \Sigma^C + (G_1 + V_2) \Sigma^E + (G_1 + C_2) \Sigma^F + (E_3 + F_{\max}) \Sigma^G \\
&\quad - (E_3 G_1 + F_{\max} G_1 + C_2 F_1) \Sigma^E .
\end{aligned}$$

#### 4.4. New groups


The groups may be extended by adding points to the polyhedron and we have carried out this procedure to extend the group  $G_2$ . A non-simply laced group results from the effect of monodromy on a simply laced group. Some of the divisors for the simply laced group are identified under the monodromy. The resulting divisor(s) of the non-simply laced group are the ones that may have  $\chi \neq 1$ . It seems that while it is frequently the case that these divisors that result from identification under monodromy have  $\chi \neq 1$  that this is not always the case. Our first extension  $G_2 \subset SO(7)$  leads to a group all of whose divisors have  $\chi = 1$ .

however as we extend the group further to  $SO(9)$ ,  $F_4$ ,  $SO(11)$  and  $SO(13)$  we find that these cases all exhibit dissident divisors. The data is summarized by the following table.

Apparent Group	$h^{11}$	$h^{31}$	$h^{22}$	$h^{21}$	$\delta$	$\bar{\delta}$	$\chi$ of dissident(s)
$G_2$	8	2897	11662	1	0	0	$\{0\}$
$SO(7)$	9	2895	11660	0	0	0	no dissidents
$SO(9)$	10	2894	11660	0	2	0	$\{3\}$
$F_4$	10	2894	11660	0	4	0	$\{3, 3\}$
$SO(10)$	11	2869	11564	0	0	0	no dissidents
$SO(11)$	11	2869	11564	0	12	0	$\{13\}$
$SO(13)$	13	2787	11244	0	25	0	$\{26\}$
$E_{7b}$	14	2790	11236	12	0	0	$\{-11\}$
$E_8$	15	2825	11292	56	0	0	$\{-55\}$

We will discuss in §6 the contribution to the superpotential from the corresponding divisors; here we make some further comments on the table:


$G_2$ :

The dissident divisor,  $E_3$  corresponds to the end of the Dynkin diagram  away from the extending root. This divisor is the one that corresponds to the three nodes of the Dynkin diagram for  $SO(8)$  that are identified under a  $\mathbb{Z}_3$  monodromy. The dissident divisor contributes one to  $h^{21}$ , and so also to  $h^{12}$  and hence contains two three-cycles.

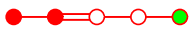
$SO(7)$ :

All the divisors have arithmetic genus unity even though the group is not simply laced.

$SO(9)$ :

There is a dissident divisor  again corresponding to the end of the Dynkin diagram away from the extending root. This is the node that corresponds to the two nodes of the  $SO(10)$  that are identified under a  $\mathbb{Z}_2$  monodromy. In this case the dissident divisor contributes to  $\delta$ , the number of non-toric parameters.

#### $F_4$ :

There are now two dissidents at the end of the Dynkin diagram  away from the extending root which correspond to the nodes of the  $E_6$  that are identified under a  $\mathbb{Z}_2$  monodromy. The dissidents contribute to  $\delta$ . The Hodge numbers of the manifolds corresponding to  $SO(9)$  and  $F_4$  are the same suggesting that they are in fact the same manifold. The number of non-toric parameters is less for  $SO(9)$  than for  $F_4$  suggesting that  $SO(9)$  gives the better description and that the true group is perhaps  $SO(8)$ .

#### $SO(10)$ :

We include this group for purposes of comparison with  $SO(11)$ . The group  $SO(10)$  is simply laced so all the nodes of the Dynkin diagram correspond to divisors with arithmetic genus unity. For this case there are no non-toric parameters.

#### $SO(11)$ :

This case is similar to  $SO(9)$ . There is a dissident divisor which contributes to  $\delta$ . The Hodge numbers for this manifold are the same as those for  $SO(10)$  which suggests that under a generic deformation the group becomes  $SO(10)$ .

#### $SO(13)$ :

This case is interesting and exhibits a new phenomenon which it has in common with the following two examples. The group is not simply laced nevertheless all the nodes of the Dynkin diagram correspond to divisors with arithmetic genus unity. The dissident divisor, in the toric description, arises not from the group per se but because the half polyhedron that projects down to a ray in the fan of  $B^X$  contains the divisor  $\{0, 0, 1, 0, 0\}$  as a point interior to a facet. This divisor is associated with a further blow-up of the base.

More precisely: after resolving the general singularities of the Weirstrass model, which gives rise to the  $S(13)$  configuration, the fourfold is still singular; the curve of singularities is a rational curve. It turns out that after one blow up the fourfold  $X$  is smooth and still satisfies the Calabi-Yau condition;  $X$  is elliptically fibered over a threefold  $B$ , which is the blow up of  $B^X$  (the common base of the other examples) along a rational curve  $\Gamma$ . The fiber of this new fibration are all curves, and there are no new gauge groups.

The elliptic threefold over the surface  $B^Y$ , obtained after resolving the general singularities over  $\sigma_\infty$  is singular at one point, after blowing up this point to another surface  $B$  and normalizing we obtain a new Calabi-Yau, mapping to  $B$ , with one dimensional fibers.

### $E_8$ :

Again the nodes of the Dynkin diagram correspond to divisors with arithmetic genus unity and the dissident divisor is as in the previous case.

A more careful statement is again that after resolving the general singularities of the Weirstrass model, which gives rise to the  $E_8$  configuration, the fourfold is still singular and further blow ups are needed. It turns out that after all the necessary blow ups the fourfold  $X$  still satisfies the Calabi-Yau condition;  $X$  is elliptically fibered over a threefold  $B$ , which is the blow up of  $B^X$  (the common base of the other examples) along a curve  $\Gamma$ . The fiber of this new fibration are all curves, and there are no new gauge groups. We can also consider the elliptic threefold over the surface  $B^Y$  ( $B^X$  is fibered by  $B^Y$ , the Calabi-Yau fourfold is fibered by such Calabi-Yau threefolds).  $B^Y$  is the Hirzebruch surface  $\mathbb{F}_3$  blown up at a point. We denote by  $f$  the exceptional divisor of this blow up,  $g$  the strict transform of a fiber of the fibration  $\mathbb{F}_3 \rightarrow \mathbb{P}^1$  and  $\sigma_\infty$  the section with negative self-intersection. Our models have a  $SU(2)$  gauge group over a section  $\sigma_0$ , such that  $\sigma_0 \cdot g = 0$ ,  $\sigma_0 \cdot f = 1$ . A quick check shows that the general elliptic fibration with an  $E_8$  gauge group over  $\sigma_\infty$ , acquires extra singularities over 9 different points  $P_i$ ,  $i = 1, \dots, 9$ , at the intersection of  $\sigma_\infty$  with the divisor of  $I_1$  singular fibers. These 9 points are the intersection of  $\Gamma$  and  $B^Y$ . It can be easily verified that the resolution of these singularities introduces 9 new divisors on the threefold: if we want to maintain the equidimensionality conditions (all the fibers being curves), then we have to blow up the base at each  $P_i$ . The new threefold is Calabi-Yau and there are no new gauge groups.

### $E_{7b}$ :

This is one of the ways of realizing  $E_7$  and is similar to the two cases above. All the nodes of the Dynkin diagram correspond to divisors with arithmetic genus unity. The dissident divisor is  $\{0, 0, 1, 0, 0\}$  which is a point interior to a facet.

## 5. The Yukawa Couplings and Mori Cone

### 5.1. The Yukawa Couplings

The topological yukawa couplings  $D_i D_j D_k D_l$  are calculated from the fan by means of the program SCHUBERT. The most important vanishing relations in the intersection ring can be read off directly from the fan or more simply from Figure 4.1. We see, for example, that  $V_1$  and  $V_6$  always lie in different cones and that  $G$  never occurs in the same cone with  $C_1$  or  $C_2$  and that  $F$  never occurs in the same cone with any of the  $E_i$ . It follows that these divisors do not intersect in  $\mathbb{P}_\nabla$  and hence do not intersect in  $X$ . In this way we learn that

$$GC_i = 0, \quad FE_j = 0, \quad C_i E_j = 0, \quad BC_2 = BE_2 = BE_3 = 0$$

$$F(F + G) = G(F + G) = 0, \quad Y^2 = 0, \quad (3H + E_3)E_2 = 0$$

where the last of these identities follows from the fact that  $V_5$  and  $E_2$  never lie in the same cone. Four further quadratic identities follow by examining the intersection numbers

$$E_1 E_3 = 0, \quad BH = 0, \quad E_1 H = 0, \quad \text{and} \quad (2H - C_2)C_1 = 0.$$

Taken together with the previous identities these furnish a basis for the quadratic relations between the divisors. The nonzero intersection numbers are given below for a slightly redundant basis that includes also the divisor  $E_3$ :

$B^4 = -82$	$B^3 C_1 = 48$	$B^3 E_1 = 4$	$B^3 F = 8$	$B^3 Y = 7$
$B^2 C_1^2 = -28$	$B^2 C_1 F = -6$	$B^2 C_1 Y = -4$	$B^2 E_1^2 = -10$	$B^2 E_1 G = 2$
$B^2 E_1 Y = 1$	$B^2 F^2 = 2$	$B^2 FG = -2$	$B^2 FY = -1$	$B^2 G^2 = 2$
$B^2 GY = -1$	$BC_1^3 = 16$	$BC_1^2 F = 4$	$BC_1^2 Y = 2$	$BC_1 FY = 1$
$BE_1^3 = 24$	$BE_1^2 G = -4$	$BE_1^2 Y = -3$	$BE_1 GY = 1$	$BF^3 = -4$
$BF^2 G = 4$	$BF^2 Y = -1$	$BFG^2 = -4$	$BFGY = 1$	$BG^3 = 4$
$BG^2 Y = -1$	$C_1^4 = -144$	$C_1^3 C_2 = 192$	$C_1^3 F = 8$	$C_1^3 Y = 8$
$C_1^2 C_2^2 = -272$	$C_1^2 C_2 F = -16$	$C_1^2 C_2 Y = -12$	$C_1^2 FY = -2$	$C_1 C_2^3 = 384$
$C_1 C_2^2 F = 24$	$C_1 C_2^2 Y = 16$	$C_1 C_2 FY = 2$	$C_2^4 = -528$	$C_2^3 F = -32$
$C_2^3 Y = -20$	$C_2^2 FY = -2$	$E_1^4 = -56$	$E_1^3 G = 8$	$E_1^3 Y = 8$
$E_1^2 E_2^2 = 2$	$E_1^2 E_2 Y = -1$	$E_1^2 GY = -2$	$E_1 E_2^3 = 4$	$E_1 E_2^2 G = -2$
$E_1 E_2^2 Y = -1$	$E_1 E_2 GY = 1$	$E_2^4 = -48$	$E_2^3 E_3 = 36$	$E_2^3 G = 8$
$E_2^3 Y = 8$	$E_2^2 E_3^2 = -18$	$E_2^2 E_3 G = -6$	$E_2^2 E_3 Y = -9$	$E_2^2 GY = -2$
$E_2 E_3^2 Y = 9$	$E_2 E_3 GY = 3$	$E_3^4 = -72$	$E_3^3 G = 24$	$E_3^2 GY = -6$

## 5.2. The Mori Cone of $X$

The Mori cone (what is this and why do we even care) of the embedding space  $\mathbb{P}_\nabla$  may be found by the method of positive piecewise linear functions as explained in KLR (see also [19]). Recall that a dimension is added to the vector space defined by the points of  $\nabla$  and a 1 is prepended to each of the points corresponding to the divisors (so that  $D_0 \sim (1, 0, 0, 0, 0)$ , etc.). The cones of the fan extend to simplicial cones whose vertex is at the origin of this extended space.

A piecewise linear function  $m$  is defined on the extended space so as to be linear on each cone. If  $u$  is a point in a cone  $\sigma$  with generators  $u_i$ ,  $i \in I$ , then  $u$  can be expressed uniquely in the form

$$\vec{u} = \sum_{i \in I} \lambda_i \vec{u}_i \quad ; \quad \lambda_i \geq 0 .$$

The function  $m(\vec{u})$  is given in terms of  $m_i = m(\vec{u}_i)$  by

$$m(\vec{u}) = \sum_{i \in I} \lambda_i m_i$$

which is equivalent to giving a vector  $\vec{m}_\sigma$  on each cone such that

$$m(\vec{u}) = \vec{m}_\sigma \cdot \vec{u} \quad \text{for} \quad \vec{u} \in \sigma .$$

The function  $m$  is a positive piecewise linear function if

$$\begin{aligned} m(\vec{u}) &= \vec{m}_\sigma \cdot \vec{u} \quad ; \quad u \in \sigma \\ m(\vec{u}) &\geq \vec{m}_\sigma \cdot \vec{u} \quad ; \quad u \notin \sigma \end{aligned} \tag{5.1}$$

The system of inequalities (5.1), being linear, is specified by the coefficients that appear. In this way, they specify a cone which is identified with the Mori cone of  $\mathbb{P}_\nabla$ . The integral basis for the system (5.1) may be identified with the generators of the Mori cone. (This process is performed in detail for the simple case of the threefold  $Z$  in appendix A.) For fourfolds this process requires computer calculation.

For the case at hand, the relations between the divisor classes may be used to express the system (5.1) entirely in terms of a basis consisting of the divisor classes

$$\{B, C_1, C_2, E_1, E_2, F, G, Y\} . \tag{5.2}$$

In other words, the  $m_i$  corresponding to the remaining divisors may be set to zero. The system (5.1) then consists of 83 inequalities which are generated by the following coefficient vectors

$$\begin{aligned}
a^{-1} &= (1, -2, 2, 0, 0, -1, 0, 0) , \\
a^0 &= (1, -1, 1, -1, 0, 0, 0, 0) , \\
a^1 &= (-1, 1, 0, 0, 0, -1, 1, 0) , \\
a^2 &= (0, 0, 0, -1, -1, 0, 1, 0) , \\
a^3 &= (-1, 0, 0, 1, 0, 1, -1, 0) , \\
a^4 &= (0, 0, 0, 1, -2, 0, 0, 0) , \\
a^5 &= (0, 0, 0, 0, 1, 0, 0, 0) , \\
a^6 &= (0, 1, -1, 0, 0, 0, 0, 0) , \\
a^7 &= (1, -3, 3, 0, 0, 0, 0, 0) , \\
a^8 &= (1, 0, 0, -2, 0, 0, 0, 0) , \\
a^9 &= (0, 0, 0, 0, -2, 0, 0, 1) ,
\end{aligned}$$

The vectors on the right are the coefficients of the inequalities that generate (5.1). We may associate them also with the curves that generate the Mori cone. The components of the vectors are also the intersection numbers

$$a^i_j = a^i \cdot D_j$$

of the curves with the divisors  $D_j$  of the basis (5.2). The Mori cone that we have obtained has nine edges  $a^i$ ,  $i = 1, \dots, 9$ . There are in addition two further generators  $a^{-1}$  and  $a^0$  which are internal to the cone and which are required because in this case the edges do not generate the cone

$$\begin{aligned}
a^0 &= \frac{1}{2}(a^6 + a^7 + a^8) , \\
a^{-1} &= \frac{1}{2}(a^0 + a^4 + a^7) .
\end{aligned}$$

Note that since there are nine edges rather than eight the cone is not simplicial. A ninth edge is however necessary since the relation of linear dependence is

$$a^1 - a^2 - 3a^3 + 4a^4 + 2a^5 = 0$$

and so no edge can be written as a positive combination of the others.

The complication is that what we have calculated is the Mori cone of the embedding space  $\mathbb{P}_\nabla$  rather than the Mori cone of  $X$ . The procedure advocated by Cox and Katz[8] is to

- (i) compute *all* the possible fans for  $\mathbb{P}_\nabla$  that is triangulate  $\nabla$  in all possible ways. The program PUNTOS can accomplish this in cases that are not too complicated.
- (ii) Compute using the program SCHUBERT the yukawa couplings  $Y^{ijkl} = D_i D_j D_k D_l$  corresponding to each fan. Fans that lead to different couplings  $Y^{ijkl}$  correspond to different phases of the theory in the sense of the linear sigma-model. Fans that lead to the same  $Y^{ijkl}$  correspond to different resolutions of the embedding space that do not affect the Calabi–Yau hypersurface  $X$ . That is the corresponding  $\mathbb{P}'_\nabla$ s are related by flopping curves that do not intersect  $X$ . Thus the fans should be grouped into classes classified by the  $Y^{ijkl}$ .
- (iii) Within a given class the Mori cones for the embedding spaces will in general be different however the true Mori cone, *i.e.*, the Mori cone for  $X$ , should be contained in each of them. Thus we proceed, for a given class, by computing the intersection of all the corresponding Mori cones of the embedding spaces.

We have carried through this program for our  $\nabla$ ; though we shall see that the resulting cone is still too large. A total of 990 fans were found which when classified by the yukawa couplings fall into 7 classes. These 7 classes turn out to correspond to the 7 ways of triangulating the  $(0, x_2, x_3, 2, 3)$  plane shown in Figure 4.1. The class corresponding to our couplings, *i.e.*, the Table at the beginning of this section, comprises 20 fans all of which correspond to the triangulation of the  $(0, x_2, x_3, 2, 3)$  plane shown in the central figure in Figure 4.1. Although they coincide in this plane these 20 fans are different leading to 20 distinct Mori cones. Of the 20 only 4 satisfy the condition that was discussed previously that each cone of the fan of  $\mathbb{P}_\nabla$  project onto some cone of the fan for  $B$ . The other fans in this class correspond to the same yukawa couplings as the fans that do satisfy the projection criterion so it must be the case that although these fans do not realize  $\mathbb{P}_\nabla$  as an elliptic fibration they do realize  $X$  as an elliptic fibration.



The task of finding the intersection of the twenty cones can be done by means of a computer program however in this particular case it is easy to do by hand. It happens that among the twenty cones there are many edges that appear in a certain cone and then with opposite sign in another cone. In such a case it is easy to see that the intersection of the two cones is contained in the cone formed by discarding the edges that appear with opposite sign and taking the union of the remaining edges for the two cones. Thus we may discard all the edges that appear with opposite sign and take the union of all the remaining edges. In this way we see that the intersection must be contained in the following cone that has the ten edges  $\ell^i$ ,  $i = 1, \dots, 10$

$$\begin{aligned}
\ell^0 &= (1, -1, 1, -1, 1, 0, 0, 0) = a^0 + a^5 \\
\ell^1 &= (0, 0, 0, -1, -1, 0, 1, 0) = E_1 E_2 Y^+ = a^2 \\
\ell^2 &= (1, -2, 2, 0, 0, 0, 0, 0) = C_1 F Y^+ = a^6 + a^7 \\
\ell^3 &= (0, 1, -1, 0, 0, 0, 0, 0) = \frac{1}{2} C_2 F Y^+ = a^6 \\
\ell^4 &= (1, 0, 0, -2, 1, 0, 0, 0) = E_1 G Y^+ = a^5 + a^8 \\
\ell^5 &= (0, 0, 0, 1, -2, 0, 0, 0) = E_2 G Y^+ = a^4 \\
\ell^6 &= (0, 0, 0, 0, 1, 0, 0, 0) = \frac{1}{3} E_3 G Y^+ = a^5 \\
\ell^7 &= (-1, 1, 0, 0, 0, -1, 1, 0) = F B Y^+ = a^1 \\
\ell^8 &= (-1, 0, 0, 1, 0, 1, -1, 0) = G B Y^+ = a^3 \\
\ell^9 &= (0, 0, 0, 0, -2, 0, 0, 1) = E_1 E_2 G = \frac{1}{3} E_2 E_3 G = a^9 \\
\ell^{10} &= (-1, 0, 1, 0, 0, 0, 0, 0) = a^0 + a^1 + a^3
\end{aligned} \tag{5.3}$$

It is easy to see also that each of the edges of this cone is contained in each of the twenty cones with which we started so that it is in fact the intersection we were seeking. Again in this case the edges do not generate the cone and we require also the internal generator

$$\ell^0 = \frac{1}{2}(\ell^2 + \ell^4 + \ell^6)$$

This cone contains the true Mori cone and by inspection we identify curves of  $X$  with each of the generators apart from  $\ell^0$  and  $\ell^{10}$ . Thus the edges  $\ell^1, \dots, \ell^9$  are true edges. Notice also that the divisors of our basis appear in the curves as

$$\{C_1, C_2\} F Y^+ = \{C_1, C_2\} Y^+ Y^- , \quad \{E_1, E_2, E_3\} G Y^+ = \{E_1, E_2, E_3\} Y^+ Y^- ,$$

$$\text{and } \{F, G\} B Y .$$

It is interesting to note that if we compute the intersection matrix between the divisors  $C_i$  and the curves  $C_j FY^+$  with  $i, j = 1, 2$  we find

$$C_i C_j FY^+ = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

which we recognise as the extended Cartan matrix for  $SU(2)$  by which we mean the matrix corresponding to including the extending root of the algebra. This is in accord with the observations of Intriligator *et al.*[20]. If we consider the intersections between the divisors  $E_i$  and the curves  $\ell^j$  with  $i = 1, 2, 3$  and  $j = 4, 5, 6$  then we find the extended Cartan matrix corresponding to  $G_2$

$$E_i \ell^j = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 3 \\ 0 & 1 & -2 \end{pmatrix}.$$

The curve  $\ell^9 = E_1 E_2 G$  maps to the  $\mathbb{P}_1$  of the base of Calabi–Yau-fibration  $X = (Y, \mathbb{P}_1)$  while  $\ell^1$  maps to a fiber of  $B^Z$ . Since there are ten edges there are two linear relations between them one of these involves  $\ell^{10}$  the other relation is more interesting and permits the elliptic fiber to be expressed in two different ways

$$\mathcal{E} = FGY = (1, 0, 0, 0, 0, 0, 0, 0) = \ell^2 + 2\ell^3 = \ell^4 + 2\ell^5 + 3\ell^6. \quad (5.4)$$

We can show that  $\ell^0$  and  $\ell^{10}$  cannot be generators of the true cone. Consider first  $\ell^{10}$ . If this is a generator of the Mori Cone then it is an irreducible curve that is contained in  $B$ , since  $\ell^{10} \cdot B = -1$ , and which also intersects  $C_2$ , since  $\ell^{10} \cdot C_2 = 1$ . But this is impossible since  $B$  and  $C_2$  do not intersect. In a similar way we see that  $\ell^0$  is not a true generator since such a curve would have to be contained in both  $C_1$  and  $E_1$  which however do not intersect.

Note now that the generators  $\ell^1, \ell^2, \ell^4, \ell^5, \ell^7, \ell^8, \ell^9$ , that is the generators  $\ell^1, \dots, \ell^9$  with  $\ell^3$  and  $\ell^6$  omitted, define a seven-plane  $L$ , say, within the eight dimensional cone. For any vector  $k$  in the lattice we can define a height relative to  $L$

$$h(k) = \det(k, \ell^1, \ell^2, \ell^4, \ell^5, \ell^7, \ell^8, \ell^9) = h \cdot k$$

where on the right  $h$  denotes the vector  $(6, 8, 5, 4, 2, 8, 6, 4)$ . Now  $h(\ell^0) = 1$ ,  $h(\ell^3) = 3$ ,  $h(\ell^6) = 2$  and  $h(\ell^{10}) = -1$  so  $\ell^{10}$  lies on one side of  $L$  and  $\ell^0, \ell^3$  and  $\ell^6$  lie on the other side. This seems to provide a counterexample to the conjecture of Cox and Katz (see also [9]) that the procedure we have followed should yield the true Mori cone.

We have seen that  $\ell^{10}$  is not a true generator and the question arises as to whether we can discard all the points that have negative height with respect to  $L$ . Now we have to express  $\ell^0$  as a positive integral combination of generators, since  $\ell^0$  cannot be a generator, and since  $h(\ell^0) = 1$  which is less than  $h(\ell^3)$  and  $h(\ell^6)$  we see that we must have at least one generator with negative height.

### 5.3. Volumes of the Divisors

There are two natural ways to parametrize the Kähler-form. The first is to write it directly in terms of the basis of divisors

$$J = tB + s_1C_1 + s_2C_2 + s_3E_1 + s_4E_2 + s_5F + s_6G + vY .$$

in this expression  $t = J \cdot \mathcal{E}$  is the volume of the elliptic fiber. Another useful parametrization of the Kähler-form is obtained by taking the volumes of the curves  $\ell^1, \dots, \ell^9$  (defined in (5.3)) as coordinates. We set

$$J \cdot \ell^i = (\mu_+, \gamma_1, \gamma_2, \epsilon_1, \epsilon_2, \epsilon_3, \delta_F, \delta_G, \mu_-) , \quad i = 1, \dots, 9 . \quad (5.5)$$

There are 9 parameters on the right so there is a linear relation which is a consequence of (5.4)

$$t = \gamma_1 + 2\gamma_2 = \epsilon_1 + 2\epsilon_2 + 3\epsilon_3 . \quad (5.6)$$

We wish now to examine the relation between the volumes of the b-divisors and the volumes of the Mori generators. This is of interest since the superpotential arises, in the M-theory description, through contributions of the form  $\exp(2\pi i \text{vol}(D))$  while in the heterotic description it arises through instanton corrections and the instantons are linear combinations of the Mori generators. Now

$$\text{vol}(D) = \frac{1}{3!} J^3 D$$

which is cubic in the parameters of  $J$  while the volumes of the Mori curves and so of course the instantons are linear in the parameters of  $J$ . The volumes of the b-divisors are for the most part complicated cubic expressions in the parameters. The simplest of these expressions are those for the volumes of  $E_1$  and  $E_2$

$$\begin{aligned} \text{vol}(E_1) &= \epsilon_1 \left( \mu_- \mu_+ + \epsilon_1 \left( \frac{1}{2} \mu_- + \mu_+ \right) + \frac{4}{3} \epsilon_1^2 \right) \\ \text{vol}(E_2) &= \epsilon_2 \mu_- \left( \mu_+ + \frac{1}{2} \epsilon_2 \right) . \end{aligned}$$

It seems that one should consider the limit with  $t$  small. Since the parameters that we have introduced through (5.5) are all positive it follows in virtue of (5.6) that the  $\gamma_i$  and

$\epsilon_j$  tend to zero with  $t$ . We expect  $\text{vol}(D)$  to tend to zero linearly with  $t$  for every b-divisor  $D$  so the neglect of terms of  $\mathcal{O}(t^2)$  leads to expressions with a term of  $\mathcal{O}(t)$  as a factor. For  $F$  we have in this limit  $\text{vol}(F) \sim t\mu_- \delta_F$ . The corresponding linearized expressions for the other b-divisors are as follows

$$\text{vol}(C_1) \sim \gamma_1(2\delta_F + 3\delta_G + \mu_+)(4\delta_F + 4\delta_G + \mu_-)$$

$$\text{vol}(C_2) \sim 2\gamma_2(2\delta_F + 3\delta_G + \mu_+)(4\delta_F + 4\delta_G + \mu_-)$$

$$\text{vol}(E_1) \sim \epsilon_1\mu_- \mu_+$$

$$\text{vol}(E_2) \sim \epsilon_2\mu_- \mu_+$$

$$\text{vol}(E_3) \sim 3\epsilon_3\mu_- \mu_+$$

$$\text{vol}(F) \sim t\delta_F(\mu_- + 2\delta_F + 4\delta_G)$$

$$\text{vol}(G) \sim t\delta_G(\mu_- + 2\delta_G) .$$

## 6. The Superpotential

### 6.1. Characterization of the Divisors Contributing to the Worldsheet Instantons

Let us denote with  $\pi : X \rightarrow B^X$  the elliptic fibration,  $p : B^X \rightarrow B^Z$  the fibration by rational fibers (generally  $\mathbb{P}^1$ ), and by  $\epsilon : X \rightarrow B^Z$  the composed K3-fibration. We assume that  $p$  is equidimensional (replacing, if necessary,  $B^Z$  by a suitable blow up). The divisors that contribute to the superpotential are the divisors  $D$  such that  $\pi(D)$  is a divisor,  $C$ , of  $B^X$  and  $p(\pi(D))$  is a curve,  $\gamma$ , in  $B^Z$ :

$$\begin{array}{ccc} X & & D \\ \downarrow \pi & \searrow \epsilon & \downarrow \pi \searrow \epsilon \\ B^X & \xrightarrow{p} & B^Z \end{array} \qquad \begin{array}{ccc} & & \\ & & \\ C & \xrightarrow{p} & \gamma \end{array}$$

Most the divisors that we construct via the toric construction contribute to space-time instantons. It turns out that divisors contributing to the worldsheet instantons are nicely divided in 3 different types, each of which has a distinct meaning in physics:

(a) If  $D$  does not correspond to a non-abelian gauge group and  $D = \epsilon^*(\gamma)$ , where  $\gamma = \epsilon(D)$ . In this case  $1 = \chi(D) = -\gamma^2$ . In fact,

$$\chi(D) = D^2 \cdot c_2/24 = -\gamma^2 \cdot \chi_{top}(S)/24 = -\gamma^2,$$

where  $S$  is the general fiber of the  $K_3$ -fibration  $X \rightarrow B^Z$ . Furthermore, the adjunction formula shows that  $\gamma$  is a smooth rational curve. Such curve is necessarily an edge of the Mori cone of  $B^Z$ .

(b) If  $D$  does not correspond to a non-abelian gauge group and  $D \neq \epsilon^*(\gamma)$ , then  $p : B^X \rightarrow B^Z$  is not a  $\mathbb{P}^1$ -bundle; not all such divisors contribute to the superpotential, even if  $\chi(\mathcal{O}_D) = 1$  (see also [21]). However the divisors  $F$  and  $G$  do contribute (see Section 4).

(c) If  $D$  corresponds to a gauge group,  $D$  arises from a degeneration of the K3 fiber  $S$ : therefore  $D$  generates a *non-perturbative* gauge group and the corresponding heterotic model will have singularities. In particular  $\epsilon(D) = \gamma$  is a component of the discriminant locus (with gauge groups) of the elliptic fibration  $Z \rightarrow B^Z$ . We have the following 2 cases:

- c.1)  $C = \pi(D) = p^*(\gamma)$
- c.2)  $C$  is a component of  $\pi(D) \subset p^*(\gamma)$

## 6.2. Comparison with the Heterotic Superpotential

At this point we do not know much about cases b) and c), so we are concentrating on the divisors of type a). Most of the explicit examples in the literature are of type a). The corresponding heterotic superpotential (via the F-theory/heterotic duality) is expected to be a function of the volume of the curves  $p(\pi(D))$  and thus to be linear. While we do not see any mathematical a priori reason of why this should be true, it turns out to be so in all the examples examined hitherto. Typically the divisors contributing to the superpotential on F-theory compactifications are finite in number: this is because  $B^X$ , the base of the elliptic fibration needs to have an effective first Chern class ( $c_1(B^X) \geq 0$ ); in the toric case also the number is always finite[22].

The examples in [23] and [22] have only a finite number of divisors and these are of type (a) and a simple computation shows that the superpotential is linear in the volumes of these divisors, up to an overall factor. The computation is more complicated in the case where there are infinitely many divisors, as in the following examples:

## 6.3. Andreas' Examples

In his paper[24] on heterotic/F-theory duality Andreas considers certain examples closely related to the one in [14]: there are infinitely many divisors contributing to the superpotential. We summarize his argument here:

The threefold  $B_n = B^X$ .

$S \rightarrow \mathbb{P}^1$  is  $\mathbb{P}^2$  blown up at 9 points and  $\mathbb{F}_r \rightarrow \mathbb{P}^1$  a Hirzebruch surface.  $B_n = B^X$  is the fiber product of  $S$  and  $\mathbb{F}_r$  with base  $\mathbb{P}^1$ , with  $2r = n$ . In particular  $B^X$  is a  $\mathbb{P}^1$ -bundle over  $S$ , but is not a product unless,  $r = 0$  (where  $B^X = S \times \mathbb{P}^1$ , as in [14]).

The threefold  $\mathcal{B}_r$ .

Let  $X_n^3 \rightarrow \mathbb{F}_n$ ,  $n = 2r$  be a smooth Calabi-Yau 3-fold, with an involution  $\tau$  compatible with the involution on  $\mathbb{P}^1$ :  $z \rightarrow -z$ . (Such threefolds can be obtained by choosing appropriate coefficients for the Weierstrass model.) Set  $X_n^3/\tau = \mathcal{B}_r$ ;  $\mathcal{B}_r$  is a smooth threefold. There is a natural elliptic fibration  $\mathcal{B}_r \rightarrow \mathbb{F}_r$ . (Note: if  $r = n = 0$ ,  $\mathcal{B}_0 = B$ , as in [14].)

The fourfold  $X_n^4 = X_r$ .

$X_r$  is the fiber product of  $S \rightarrow \mathbb{P}^1$  and  $\mathcal{B}_r \rightarrow \mathbb{P}^1$ . By construction  $X_r$  is elliptically fibered over  $B^X$ , while is fibered by K3 surfaces over  $S = B^Z$  (the basis of the heterotic dual).

It can be verified that  $X_r$  is a Calabi-Yau 4-fold, with heterotic dual  $X_n^3$ . (If  $r = 0$ , then  $X_0$  is the Weierstrass model of the  $X$  in [14]; the calculation in [14] computes also the superpotential for  $X_0$ , up to a factor.) We are interested in the contribution to the superpotential from worldsheet instantons.

The divisors contributing to the superpotential

The divisors contributing to the superpotential via worldsheet instanton are, as in [14] the inverse images of the section of the fibration  $S \rightarrow \mathbb{P}^1$ . As in [14], they are all isomorphic to  $\mathcal{B}_r$ . If  $r > 0$ , there are other divisors contributing to the superpotential via spacetime instantons (some correspond to gauge groups).

Denote by  $\{\Gamma_0, \Gamma_1, \dots, \Gamma_s\}$  the generators of  $H^2(B^X, \mathbb{R})$  and by

$$\{B, \pi^*(\Gamma_0), \dots, \pi^*(\Gamma_s), \Lambda_1, \dots, \Lambda_w\}$$

the generators  $H^2(X, \mathbb{R})$ , where  $B \sim B^X$  and the  $\Lambda_j$  correspond to gauge groups (there are the exceptional divisors of the morphism to the Weierstrass model). Without loss of generality we take  $\Lambda_j \cdot B = 0$  and take  $\Gamma_0 \sim S \sim B^Z$  and  $\Gamma_i = p^*(\gamma_j)$ , where  $\{\gamma_j\}$  generate the Kähler cone of  $S$ . As before we can express the Kähler form as a linear combination of the divisors

$$J = tB + \tilde{\Gamma}(u) + \Lambda(v), \quad \text{with} \quad \tilde{\Gamma}(u) = \sum_j u^j \pi^*(\Gamma_j) \quad \text{and} \quad \Lambda(v) = \sum_k v^k \Lambda_k.$$

The volume form for a threefold is then

$$\frac{1}{3!} J^3 = \frac{1}{3!} \left( tB + \tilde{\Gamma}(u) + \Lambda(v) \right)^3$$

We will now compute the volume of a *worldsheet instanton*, that is,  $\text{vol}(D_\gamma)$  for  $D_\gamma = \pi^*(p^*\gamma)$ , with  $\gamma$  a section of the elliptic fibration  $B^Z \rightarrow \mathbb{P}^1$ . We use the following facts:  $B^2 = B|_B = K_B$ , and  $-K_{B^Z}$  is a fiber of the elliptic fibration  $B^Z \rightarrow \mathbb{P}^1$ . By the adjunction formula  $B^2 = K_B = -2S - (r+1)p^*(F)$ , where  $F$  is a fiber of the fibration  $S \rightarrow \mathbb{P}^1$  and  $B^3 = K_B \cdot K_{B|B}$ . Note that again we have  $t = \text{vol}\mathcal{E}$ , and that, if  $f$  is the homology class of the  $\mathbb{P}^1$  bundle  $p: B^X \rightarrow S$ ,  $t_0 = (\sum t_j \pi^* \Gamma_j) \cdot f = S \cdot f = \text{vol}f$ . Using the geometry of

the fiber products involved we see that:

$$\begin{aligned}
B^2 \cdot \tilde{\Gamma}(u) \cdot D_\gamma &= -(r-1)\text{vol}_B(f) - 2\text{vol}_S(\gamma) \\
B^3 D &= 4 \\
\tilde{\Gamma}(u)^3 \cdot D_\gamma &= 0 \\
\Lambda(v)^3 \cdot D_\gamma &= d(r, v) \\
B^2 \cdot \Lambda(v) \cdot D_\gamma &= 0 \\
\Lambda(v) \cdot \tilde{\Gamma}(u)^2 \cdot D_\gamma &= 0 \\
B \cdot \Lambda(v)^2 \cdot D_\gamma &= 0 \\
\tilde{\Gamma}(u) \cdot \Lambda(v)^2 \cdot D_\gamma &= c(r, u, v) \text{vol}_{B^X}(f) + c'(r, u, v) \text{vol}_S(\gamma) \\
\tilde{\Gamma}(u)^2 \cdot B \cdot D_\gamma &= 2\text{vol}_S(\gamma) \text{vol}_{B^X}(f) - r \text{vol}_{B^X}^2(f) \\
B \cdot \Lambda(v) \cdot \tilde{\Gamma}(u) &= 0,
\end{aligned}$$

where  $d(r, v), c(r, u, v), c'(r, u, v)$  are linear function on  $r$ , which do not depend on the choice of  $\gamma$ , and are zero for  $r = 0$ . Then:

$$\exp(-\text{vol}(D)) = \exp(A) \times \exp[-C \text{vol}(\gamma)],$$

where  $A$  and  $C$  are the same for every divisor  $D_\gamma$ . This function, up to a constant depending only on  $r$ , is the expression in [14] (and for  $r = 0$  is equal to this expression); then, up to a constant the superpotential is as in [14].

#### 6.4. Comparison with $d = 3$ Dimensional Yang-Mills Theory

Katz and Vafa in [2,3] consider divisors contributing to the superpotential arising from resolution of singularities of the Weierstrass model; the divisors are associated to a simple gauge group  $G$ . They assume that there is no adjoint matter. By the chain of duality in [3],  $F$ -theory compactified on a circle is dual to  $M$ -theory compactified on the Calabi-Yau manifold  $X$ ; the radius of the circle is the inverse to the Kähler class of the elliptic fiber. In this way, one obtains a  $N = 2$  theory with  $d = 3$ ; Katz and Vafa show that, under this duality the  $F$ -theory superpotentials become the expected superpotential. A key point in their computation is that each divisor  $D_i$  corresponding to the nodes of the affine Dynkin



diagram of the Group  $G$  contribute to the superpotential, that is they satisfy the conditions (2.4) and (2.5). For examples, in the simply laced cases, one can write  $[\mathcal{E}] = \sum_{i=1}^{r+1} a_i [e_i]$ , where  $\mathcal{E}$  is the class of the elliptic fiber,  $e_i$  is the fiber of each  $D_i$  (a ruled surface) and  $a_i$  is the Dynkin index of the corresponding node of the Dynkin diagram).

### 6.5. New Features for a Non-Simply Laced Group

While the argument implied by the chain of dualities should imply the same conclusion, we are unable to make the argument work for the cases where not all the divisors satisfy the condition that  $\chi(\mathcal{O}_D) = 1$ , as in the example in Sections 5 and 6. These mixed configurations, in which some but not all of the nodes of the Dynkin diagram contribute to the superpotential, correspond to a genuine instability since in these cases it is not possible to satisfy the conditions  $dW = 0$  corresponding to the supersymmetric vacuum states.

If  $\chi(\mathcal{O}_D) > 1$ , then  $h^2(D) > 0$ . It follows that  $D$  is not general in the moduli of  $X$ , that is the locus for which  $D$  deforms in the family  $\mathcal{X}$  (of complex deformation of  $X$ ) is a complex submanifold of codimension  $h^2(D)$ . The argument needed is a modification of the corresponding statement for Calabi-Yau threefolds in [25]. In this case  $h^2(D)$  contributes to the number of non-toric parameters. We will argue in this case that the Calabi-Yau fourfold is non-general and the usual techniques for counting the divisors contributing to the superpotential do not suffice. For example, there is evidence that one should also consider the contribution of reducible divisors.

If  $\chi(\mathcal{O}_D) \leq 0$ , then  $h^1 > 0$ ; in the our examples  $h^2(D) = 0$  and there are no non-toric parameter, so the divisor  $D$  will be effective (with the same Hodge number), for all points of the complex moduli space of  $X$ . Following [12] we see that in the toric case

$$h^{2,1}(X) = \sum_{q \in \dim \tilde{\theta} = 2} (1 - \chi_q) = \sum h^1(D) .$$

This gives an interesting, yet-little studied structure on the heterotic dual [26].

By construction the Calabi-Yau fourfold  $X$  is fibered by the family of Calabi-Yau threefolds  $Y$ . It follows that the locus for which  $D_Y$  deforms in the family  $\mathcal{Y}$  (of complex deformation of  $Y$ ) is a complex submanifold of codimension  $h^1(D_Y)$ . At this point we are not sure of the implication of this fact.

### 6.6. *The prefactor.*

Ganor [27] argued that the contribution of a divisor  $D$  to the superpotential is multiplied by a pre-factor  $f$ . In most cases, the pre-factor is non-zero; Ganor gives a necessary and sufficient condition for this prefactor to vanish. Interestingly this can happen only when  $h^{2,1}(X) > 0$ . So one would hope that in the Dynkin diagram configurations with “dissident divisors” the prefactor would actually be zero for the non dissident ones. An easy computation in the case of  $G_2$ , §4.4, shows that the prefactor is nevertheless non-zero for the divisors contributing to the superpotential.

## Appendix A: Geometry of $Z$

### A.1. The Divisors

The polyhedron that we obtain by deleting the third column of  $\nabla^X$  has the structure

Relation to vertices	$\chi$	$\nabla^Z$	Divisor
$v_1$	0	(-1, 0, 2, 3)	$K_+$
$v_2$	0	( 0,-1, 2, 3)	$K_-$
$v_3$	-24	( 0, 0,-1, 0)	$2H^Z$
$v_4$	-57	( 0, 0, 0,-1)	$3H^Z$
$\frac{1}{2}(v_1+v_6) = \frac{1}{2}(v_2+v_5)$	1	( 0, 0, 2, 3)	$B^Z$
$v_5$	0	( 0, 1, 2, 3)	$K_-$
$v_6$	0	( 1, 2, 2, 3)	$K_+$
$(h^{11}, h^{21}) = (3, 243) , \quad \chi_E = -480 , \quad (\delta, \tilde{\delta}) = (0, 0)$ $H^Z = B^Z + 2K_+ + 2K_-$			

The manifold  $Z$  is elliptically fibered over a base  $B^Z = \mathbb{P}_1 \times \mathbb{P}_1$ . We denote these two  $\mathbb{P}_1$ 's by  $\mathcal{L}_+$  and  $\mathcal{L}_-$  and we see that the elliptic fibers over  $\mathcal{L}_\pm$  form K3-surfaces

$$K_\pm = (\mathcal{E}, \mathcal{L}_\pm) .$$

The fan for  $Z$  is

$$\Sigma^Z = (v_1 + v_6)(v_2 + v_5)\Sigma^\mathcal{E} , \quad \text{with} \quad \Sigma^\mathcal{E} = B^Z(v_3 + v_4) + v_3v_4 .$$

Given the fan SCHUBERT immediately provides the intersection numbers:

$$(B^Z)^3 = 8 , \quad (B^Z)^2.K_\pm = -2 , \quad B^Z K_+ K_- = 1 , \quad K_\pm^2 = 0 .$$

Finding the Kähler and Mori cones of  $Z$  is an easy exercise in the method of piecewise linear functions (though more elementary procedures work in this case also). In a notation analogous to that introduced in Sect. 5.2 one is led to the following inequalities

$$\begin{aligned}
0 &\leq -2m_b + m_1 + m_6 \\
0 &\leq -2m_b + m_2 + m_5 \\
0 &\leq m_b - 6m_0 + 2m_3 + 3m_4 \\
0 &\leq -12m_0 + m_1 + 4m_3 + 6m_4 + m_6 \\
0 &\leq -12m_0 + m_2 + 4m_3 + 6m_4 + m_5 .
\end{aligned}$$

A basis is provided by the first three inequalities so taking a basis of divisors to be  $(B^Z, v_6, v_5) = (B^Z, K_+, K_-)$  we see that the curves that generate the Mori cone are

$$\begin{aligned}
&(-2, \quad 1, \quad 0) \\
&(-2, \quad 0, \quad 1) \\
&(\quad 1, \quad 0, \quad 0)
\end{aligned}$$

which are the curves

$$\mathcal{L}_+ = B^Z K_+ , \quad \mathcal{L}_- = B^Z K_- , \quad \mathcal{E} = K_+ K_- .$$

The generators of the Kähler cone are the divisors  $(K_-, K_+, H^Z)$  that are dual to these curves.

We may write the Kähler-form as a linear combination of the generators

$$J^Z = tH^Z + u_- K_+ + u_+ K_- .$$

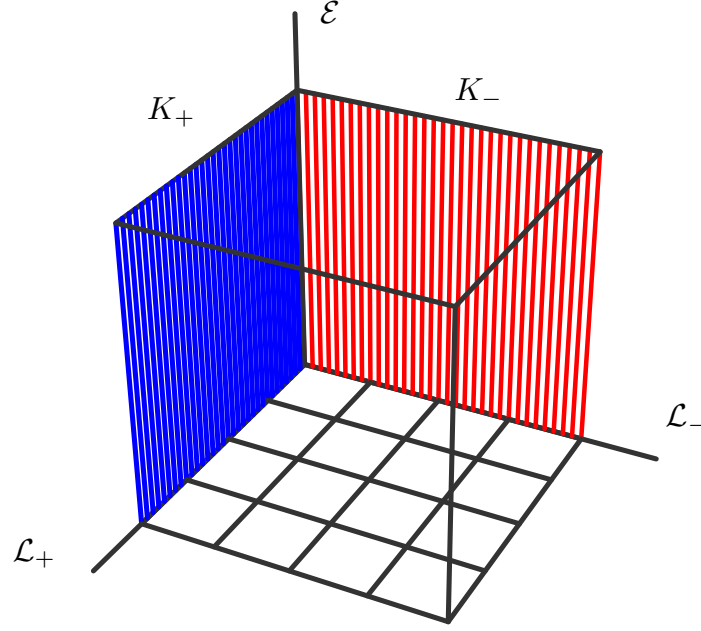
Written this way the parameters are the volumes of the dual curves:

$$J^Z \mathcal{E} = t , \quad J^Z \mathcal{L}_\pm = u_\pm .$$

We record here also the volume of  $Z$  itself as well as that of the base  $B^Z$  and that of the  $K_\pm$ .

$$\begin{aligned}
\frac{1}{3!}(J^Z)^3 &= t \left[ u_+ u_- + t(u_+ + u_-) + \frac{4}{3}t^2 \right] \\
\frac{1}{2!}(J^Z)^2 B^Z &= u_+ u_- , \quad \frac{1}{2!}(J^Z)^2 K_\pm = t(u_\pm + t) .
\end{aligned}$$

Note that, owing to the fact that the elliptic fiber varies over the base, the volume of  $Z$  is not simply the volume of the base multiplied by the volume of the fiber unless  $t$  is small.



**Figure A.1:** A sketch of  $Z$  as an elliptic fibration over  $\mathcal{L}_+ \times \mathcal{L}_-$  and as a  $K3$ -fibration in two ways as  $(K_+, \mathcal{L}_-)$  and  $(K_-, \mathcal{L}_+)$ .

### A.2. Projection to $B^Z$

As discussed in Section 3 the projection to  $Z$  corresponds to projecting out the third component of the points of  $\nabla^X$ . In reality the projection to  $Z$  exists in only a limited sense. There is a well defined projection to  $B^Z$  which is a section of the fibration  $B = (\mathbb{P}_1, B^Z)$ . The section is not unique nevertheless there are projections onto each of these sections.

We may also project the divisors of  $X$  onto the divisors of  $Z$ . This proceeds in the following way. The projection of  $C_2 \simeq (0, 0, -1, 1, 2)$  is  $(0, 0, 1, 2)$  which is interior to a codimension one face of  $\nabla^Z$ . We therefore take  $C_2$  to project to zero. In an analogous way we see that we should also take  $E_3$  to project to zero. Now we see from the polyhedra that we should take

$$H \rightarrow H^Z, \text{ and } Y^\pm \rightarrow K_\pm.$$

Now  $H = B + C_1 + C_2 + E_1 + E_2 + 2Y^+ + 2Y^-$  and  $H^Z = B^Z + 2K_+ + 2K_-$ . So we take

also

$$B \rightarrow B^Z \text{ and } C_i \rightarrow 0, \ E_j \rightarrow 0 \text{ for all } i, j.$$

Now there is an element of choice in what we wish to call the preimage of  $B^Z$  under this projection since we are free to add multiples of the divisors that project to zero. For the intersection calculation that follows it is sufficient to take this preimage to be  $\hat{B} = B + C_1 + E_1$ . We check that we obtain the correct values for the intersection numbers on  $Z$ :

$$\begin{aligned} (B^Z)^3 &= \hat{B}^2 B^Z = \hat{B}^2 B E_1 = 8 \\ (B^Z)^2 K_{\pm} &= \hat{B} Y^{\pm} B^Z = \hat{B} Y^{\pm} B E_1 = -2 \\ B^Z K_+ K_- &= Y^+ Y^- B^Z = Y^+ Y^- B E_1 = 1. \end{aligned}$$

Note that we could take instead  $B^Z = B C_1$  and these intersection numbers would still be correct. Now observe that since the images of  $F$  and  $G$  under the projection are both multiples of  $K_-$  we should set

$$F \rightarrow \alpha K_- \text{ and } G \rightarrow (1 - \alpha) K_-$$

for some  $\alpha$ . It turns out that  $\alpha$  is 0 or 1 depending on whether we take  $B^Z$  to be  $B E_1$  or  $B C_1$  since in the first case  $B^Z$  intersects  $G$  but not  $F$  while in the second  $B^Z$  intersects  $F$  but not  $G$ .

## Appendix B: The Divisors for the Spaces $Y^\pm$

We record here Tables for the divisors of  $Y^+$  and  $Y^-$ . It is evident from the topological numbers that the two manifolds  $Y^\pm$  are different.

Relation to vertices	$\chi$	$\nabla^{Y^+}$	Divisor
$V_2^+$	0	(-1, 0, 2, 3)	$Y^+ = F^+ + G^+$
$V_3^+$	1	( 0,-1, 2, 3)	$C_1^+$
$V_9^+$	1	( 0,-1, 1, 2)	$C_2^+$
$V_4^+$	-13	( 0, 0,-1, 0)	$2H^+ + E_3^+ - C_2^+$
$V_5^+$	-38	( 0, 0, 0,-1)	$3H^+ + E_3^+ - C_2^+$
$\frac{1}{2}(V_3^+ + E_1^+)$	1	( 0, 0, 2, 3)	$B^+$
$\frac{1}{2}(V_2^+ + V_F)$	1	( 0, 1, 2, 3)	$E_1^+$
$V_8^+$	1	( 0, 2, 2, 3)	$E_2^+$
$\frac{1}{2}(V_5^+ + V_8^+)$	2	( 0, 1, 1, 1)	$E_3^+ = C_1^+ + C_2^+ - (E_1^+ + 2E_2^+ + 2F^+ + 3G^+)$
$V_F$	1	( 1, 2, 2, 3)	$F^+$
$V_7^+$	1	( 1, 3, 2, 3)	$G^+$
$H^+ = B^+ + C_1^+ + C_2^+ + E_1^+ + E_2^+ + 2Y^+$ $(h^{11}, h^{21}) = (7, 169) , \quad (\delta, \tilde{\delta}) = (0, 1) , \quad \chi_E = -324$			

Table B1: The divisors for  $Y^+$ .

Relation to vertices	$\chi$	$\nabla^{Y^-}$	Divisor
$V_1^-$	0	(-1, 0, 2, 3)	$Y^-$
$V_3^-$	1	( 0,-1, 2, 3)	$C_1^-$
$V_9^-$	1	( 0,-1, 1, 2)	$C_2^-$
$V_4^-$	-14	( 0, 0,-1, 0)	$2H^- + E_3^- - C_2^-$
$V_5^-$	-45	( 0, 0, 0,-1)	$3H^- + E_3^- - C_2^-$
$\frac{1}{3}(2V_3^- + E_2^-)$	1	( 0, 0, 2, 3)	$B^-$
$\frac{1}{2}(B^- + E_2^-)$	1	( 0, 1, 2, 3)	$E_1^-$
$\frac{1}{2}(V_1^- + V_6^-)$	1	( 0, 2, 2, 3)	$E_2^-$
$\frac{1}{2}(V_5^- + E_2^-)$	-1	( 0, 1, 1, 1)	$E_3^- = C_1^- + C_2^- - (E_1^- + 2E_2^- + 4Y^-)$
$V_6^-$	0	( 1, 4, 2, 3)	$Y^-$
$H^- = B^- + C_1^- + C_2^- + E_1^- + E_2^- + 2Y^-$ $(h^{11}, h^{21}) = (8, 194) , \quad (\delta, \tilde{\delta}) = (0, 2) , \quad \chi_E = -372$			

Table B2: The divisors for  $Y^-$ .



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